

ON EXPLICIT LOCAL SOLUTIONS OF ITÔ DIFFUSIONS

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ABSTRACT. Strong solutions of p -dimensional stochastic differential equations $dX_t = b(X_t, t)dt + \sigma(X_t, t)dW_t$, $X_s = x$ that can be represented locally in *explicit simulation* form $X_t = \phi\left(\int_s^t U_{s,u}dW_u, t\right)$ are considered. Here; W is a multidimensional Brownian motion; U, ϕ are continuous functions; and b, σ, ϕ are locally continuously differentiable. The following three-way equivalence is established: 1) There exists such a representation from all starting points (x, s) , 2) U, ϕ satisfies a set differential equations, and 3) b, σ satisfy commutation relations. Next, construction theorems, based on a diffeomorphism between the solutions X and the strong solutions to a simpler Itô integral equation, with a possible deterministic component, are given. Finally, motivating examples are provided and reference to its importance in filtering and option pricing is given.

1. INTRODUCTION

Inasmuch as computability can be of utmost importance, one often confines selection of stochastic differential equation (SDE) models to those facilitating calculation and simulation. This is perhaps best exemplified in mathematical finance, where the popularity of the inaccurate Black-Scholes model is only justifiable through the evaluation ease of the resulting derivative product formulae. Indeed, Kunita (1984, p. 272) writes in his notes on SDEs that “It is an important problem in applications that we can compute the output from the input explicitly”. We shall call such solutions *explicit solutions*.

Filtering applications (see Kouritzin (1998)), option pricing applications (see companion paper Kouritzin (2016)) and pedagogical considerations initially prompted our classifications of which Itô processes $X_t^{x,s}$, starting at (x, s) , are representable as a time-dependent function of a simple stochastic integral $\phi^{x,s}\left(\int_s^t U_{s,u}dW_u, t\right)$. However, our determination of $\phi^{x,s}$, $U_{s,u}$ also facilitates an effective means of calculation and simulation. To simulate, one merely needs to compute the Gauss-Markov process $\int_s^t U_{s,u}dW_u$ at discrete times and substitute these samples into $\phi^{x,s}$, which is often known in closed form and otherwise

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is the solution of differential equations that can be solved numerically a priori. The idea is applied to strong solutions here and extended to weak solutions of a popular financial model in Kouritzin (2016). $\int_s^t U_{s,u} dW_u = \int_s^t U_{s,u}(X_u) dW_u$ can depend upon X but not in a way that will destroy its Gaussian distribution nor make simulation difficult and our explicit solutions are diffusion solutions for all starting points (x, s) . This Explicit Solution Simulation is without (Euler or Milstein) bias and is extremely efficient, often orders of magnitude faster than Euler or Milstein methods when our method is applicable and high accuracy is desired (see Kouritzin (2016)). Our representations also make properties of certain stochastic differential equations readily discernible and simplifies some filtering calculations. Finally, as demonstrated in Karatzas and Shreve (1986, Proposition 5.2.24), explicit solutions can be useful in establishing convergence for solutions of stochastic differential equations.

Doss (1977) and Sussmann (1978) were apparently the first to solve stochastic differential equations through use of differential equations. In the multidimensional setting, Doss imposed the Abelian condition on the Lie algebra generated by the vector fields of coefficients and showed, in this case, that strong solutions, X_t^x , of Fisk-Stratonovich equations are representable as $X_t^x = \rho(\Phi(x, W)_t, W_t)$, for some continuous ρ, Φ solving differential equations. Under the restriction of C^∞ coefficients, Yamato (1979) extended the work of Doss by dispensing with the Abelian assumption in favour of less restrictive q step nilpotency, whilst also introducing a simpler form for his explicit solutions $X_t^x = u(x, t, (W_t^I)_{I \in F})$. Here, u solves a differential equation, and $(W_t^I)_{I \in F}$ are iterated Stratonovich integrals with integrands and integrators selected from (t, W_t^1, \dots, W_t^d) . Another substantial work on explicit solutions to stochastic differential equations is due to Kunita (1984)[Section III.3]. He considers representing solutions to time-homogeneous Fisk-Stratonovich equations via flows generated by the coefficients of the equation under a commutative condition similar to ours, and, more generally, under solvability of the underlying Lie algebra. Kunita's work therefore generalizes Yamato (1979). Perhaps, the two most distinguishing features of our work are: We allow time-dependent coefficients and utilize a different representation that is very useful in simulation and other applications (see e.g. Kouritzin (1998), Kouritzin (2016)). We compare our results to Yamato (1979) and Kunita (1984) in Section 4.

In order to describe our method, we mention that the hitherto rather ad hoc, state-space diffeomorphism mapping method has been used to construct solutions to interesting stochastic differential equations from solutions to simpler ones. The idea of this method is to change the infinitesimal generator L of a simple Itô process to the generator \mathcal{L} corresponding to a more complicated Itô process via $\mathcal{L}f(x) = \{L(f \circ \Lambda^{-1})\} \circ \Lambda(x)$. This corresponds to using Itô's formula on $X_t = \Lambda^{-1}(\xi_t)$, where ξ is a diffusion process with infinitesimal generator L . For related examples, we refer the reader to the problems in Friedman (2006)[page 126] or Ethier and Kurtz (1986)[page 303].

Motivated by applications in filtering, Kouritzin and Li (2000) and Kouritzin (2000) used differential equation methods to study: “When can global, time-dependent diffeomorphisms be used to construct solutions to Itô equations?”, “What scalar Itô equations can be solved via diffeomorphisms?”, and “How can one construct these diffeomorphisms?”. They considered scalar solutions in an open interval D to the time-homogeneous stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x, \quad (1.1)$$

which are of the form $\phi^x \left(\int_0^t U_u dW_u, t \right)$, and showed that all nonsingular solutions of this form were actually (time-dependent) diffeomorphisms $\Lambda_t^{-1}(\xi_t)$ with ξ satisfying

$$d\xi_t = (\chi - \kappa\xi_t)dt + dW_t, \quad \xi_0 = \Lambda_0(x).$$

Nonsingular in this scalar case was interpreted as finiteness of $\int_\lambda^y \sigma^{-1}(x)dx$ for some fixed point λ and all $y \in D$. (Their methods involve non-stochastic differential equations that can continue to hold in the singular situations when global diffeomorphisms fail.)

For our current work, we suppose henceforth that $D \subset \mathbb{R}^p$ is a bounded convex domain, $T > 0$, and define

$$D_T = \begin{cases} D & \text{if } \sigma, b \text{ do not depend on } t \\ D \times [0, T) & \text{if either do} \end{cases}$$

so $(x, s) \in D_T$ means $x \in D$ when $D_T = D$. Then, we resolve the question: “When can we explicitly solve vector-valued Itô equations

$$dX_t = b(X_t, t)dt + \sigma(X_t, t)dW_t, \quad X_s = x, \quad (1.2)$$

with the dimensions of X_t, W_t being p, d respectively, through representations of the form $X_t^{x,s} = \phi^{x,s} \left(\int_s^t U_{s,u} dW_u, t \right)$?. This question is more precisely broken into two separate important questions: “For which σ and b does such a strong-local-solution representation exist?” and “What conditions are required on ϕ and U for such representations with $\int_s^t U_{s,u} dW_u = \int_s^t U_{s,u}(X_u) dW_u$ still being Gauss-Markov?” Equivalently, we consider “When can the solutions to the Fisk-Stratonovich equation

$$dX_t^x = h(X_t^x, t)dt + \sigma(X_t^x, t) \bullet dW_t, \quad (1.3)$$

with

$$h = b - \frac{1}{2} \sum_{j=1}^d \{ \nabla_\varphi \sigma_j \} \sigma_j \text{ on } D_T \quad (1.4)$$

and σ_j denoting the j^{th} column of the matrix σ , be locally represented in this manner?” It follows from, for example, Kunita (1984)[p. 239] that the unique local solutions to these (1.2) and (1.3) are equal if (1.4) holds and σ is twice continuously differentiable or satisfies the Fisk-Stratonovich acceptable condition in D , the latter being discussed in Protter (2004)[Chapter 5]. We work

with Itô equations to avoid these stronger assumptions on σ but still relate b and h through (1.4). Also, to obtain simple, concrete necessary and sufficient conditions for such a representation, we consider all solutions starting from each $(x, s) \in D_T$. Actually, assuming natural regularity conditions and using differential form techniques, we obtain very satisfying answers to these question by showing the equivalence of the following three conditions: 1) The SDEs (1.2) have our local-solution-representations for all starting points $(x, s) \in D_T$. 2) The representation pair $\phi^{x,s}, U_{s,t}$ satisfy a system of differential equations. 3) The SDE coefficients σ and h satisfy simple commutator conditions. In the process of establishing this three-way equivalence, we also answer the question “When is (1.2) locally diffeomorphic to an SDE with a simple diffusion coefficient?” i.e. “When will it have a representation as in (1.5,1.6) to follow?”. It turns out that this representation facilitates explicit weak solution of the important financial Heston model as is shown in Kouritzin (2016).

Given precise conditions of when an Itô equation has such a representation, the next natural questions we answer are: “What form do the solutions have?” and “How do you construct such solutions?” In order to include as many interesting examples as possible we will only require *local* representation $X_t^{x,s} = \phi^{x,s} \left(\int_s^t U_{s,u} dW_u, t \right)$ and allow σ to have rank less than $\min(p, d)$. The first opportunity borne out of allowing the rank of $\sigma(x)$ to be less than p is the ability to handle time-dependent coefficients, treating time as an extra state. The second advantage from allowing lesser rank than $\min(p, d)$ is the extra richness afforded by appending a deterministic equation into the diffeomorphism solution. A third, important benefit of this general rank condition is the possibility of producing explicit *weak* solutions to SDEs where no explicit strong solution exists (see Kouritzin (2016)). In our construction results, we show that ϕ is constructed via a time-dependent diffeomorphism Λ_t , which in turn is defined in terms of σ . The diffeomorphism separates a representable SDEs into deterministic and stochastic differential equations: $\Lambda_t(X_t) = (\bar{X}_t, \tilde{X}_t)$, where $\tilde{X}_t \in \mathbb{R}^{p-r}$ is deterministic and satisfies the differential equation

$$\frac{d}{dt} \tilde{X}_t = \tilde{h}(\tilde{X}_t, t), \quad (1.5)$$

while \bar{X}_t is a Gauss-Markov process satisfying

$$d\bar{X}_t = (\bar{\theta}(\tilde{X}_t, t) + \bar{\beta}(\tilde{X}_t, t)\bar{X}_t)dt + \left(I_r \middle| \bar{\kappa}(\tilde{X}_t, t) \right) dW_t. \quad (1.6)$$

$\bar{\kappa}$ is determined (within an equivalence class) by σ while $\bar{\theta}, \tilde{h}$ and $\bar{\beta}$ can be anything (subject to dimensional and differentiability regularity conditions). These parameters allow us to handle a whole class of *nonlinear* drift coefficients b for a given σ in the SDE (1.2) for $X_t = \Lambda_t^{-1}(\bar{X}_t, \tilde{X}_t)$.

In the next section, we introduce notation and state the main existence results. In Section 3, we build off of these existence results to give our construction results, illustrated with simple applications. We compare our work

to prior work of Yamato and Kunita in Section 4. The proofs of all main results are postponed to Section 5.

2. NOTATION AND EXISTENCE RESULTS

Let $(W_t)_{t \geq 0}$ be a standard d -dimensional Brownian motion with respect to filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual hypotheses on a complete probability space (Ω, \mathcal{F}, P) . We will use ϕ to denote a representation function and x to denote a starting point as in the introduction. On the other hand, φ will denote a variable with the same dimension p as ϕ and x .

For functions of time or paths of a stochastic process, we use Z_t and $Z(t)$ interchangeably. For a matrix V , V_j will denote its j^{th} column vector and $V_{i,j}$ the i^{th} element of this j^{th} column.

$B_z(\delta)$ denotes an open Euclidean ball centered at z with radius $\delta > 0$. Suppose $m, r \in \mathbb{N}$, $O \subset \mathbb{R}^m$ is open and $I \subset [0, T]$ is an interval. Then, $C(I)$ is the continuous functions on I and $C^r(O)$ denotes the continuous functions whose partial derivatives up to order r exist and are continuous on O . Moreover, $C^{r,1}(O \times I)$ denotes the continuous functions $g(\varphi, t)$ whose mixed partial derivatives in $\varphi \in O$ up to order r and in $t \in I$ up to order 1 all exist and are continuous functions on $O \times I$. $C^1(O \times I) = C^{1,0}(O \times I) \cap C^{0,1}(O \times I)$. (We only require one-sided derivatives in time to exist at interval endpoints.) For such functions of both φ and t , $\nabla_\varphi g$ is the Jacobian matrix of vector function g , that is $(\nabla_\varphi g)_{i,j} = \partial_{\varphi_j} g_i$, while ∇g will include the time derivative as the last column.

The purpose of our representations is to simulate a large class of processes in an efficient manner, which leads to a dilemma. We would like to allow $U_{s,t}$ to depend upon $X^{x,s}$ for generality but not in a way that would destroy the ease of simulation. Our approach to this dilemma is to allow $U_{s,t}$ to act as an operator on the functions $\phi^{x,s}(y_u, u)|_{u \in [s,t]}$ but then impose the condition that the result $U_{s,t}\phi^{x,s}(y, \cdot)$ can not depend upon y . As we will expose below, this basically allows $U_{s,t}$ to depend upon some hidden deterministic part of X but not the purely stochastic part, saving the Gaussian nature of

$$Y_t^s = \int_s^t U_{s,u} \phi(Y_u^s, \cdot) dW(u) = \int_s^t U_{s,u} \phi(0, \cdot) dW(u) \quad (2.1)$$

so it can be computed off-line, which is the point of this work. Then, ϕ must be differentiable enough to apply Itô's formula and allow room for random process Y_t^s to move. Finally, we want $U_{s,t}\phi$ to satisfy some type of simple state equation so it is easy to compute. The precise regularity conditions for potential representations $X_t^{x,s} = \phi^{x,s}(Y_t^s, t)$ now follow:

- \mathcal{C}_1 : For each $(x, s) \in D_T$, there is a $t_0 = t_0^{x,s} > s$ and a convex neighbourhood $\mathcal{N}^{x,s} \subset \mathbb{R}^d$ of 0 such that $\phi^{x,s} \in C^{2,1}(\mathcal{N}^{x,s} \times [s, t_0]; \mathbb{R}^p)$ and $t \rightarrow U_{s,t}\phi^{x,s}(y, \cdot) \in C^1([s, t_0]; \mathbb{R}^{d \times d})$.

\mathcal{C}_2 : $\phi^{x,s}, U_{s,t}$ start correctly

$$\phi^{x,s}(0, s) = x, \quad U_{s,s}\phi^{x,s}(0, s) = I_d \quad \forall (x, s) \in D_T. \quad (2.2)$$

\mathcal{C}_3 : $U_{s,t}\phi^{x,s}$ is non-singular on $\mathcal{N}^{x,s} \times [s, t_0)$ (with matrix inverse denoted $U_{s,t}^{-1}\phi^{x,s}$) and satisfies

$$U_{s,t}\phi^{x,s}(y_u, u) = U_{s,t}\phi^{x,s}(0, u) \quad (2.3)$$

as well as

$$U_{s,t}^{-1}\phi^{x,s}(y_t, t) \frac{d}{dt} U_{s,t}\phi^{x,s}(y_u, u) \Big|_{u=t} = \frac{d}{dt} U_{u,t}\phi^{\phi^{x,s}(y_u, u), u}(y_u, u) \Big|_{u=t}. \quad (2.4)$$

Then, (2.2, 2.4) imply

$$U_{s,t}^{-1}\phi^{x,s}(y_t, t) \frac{d}{dt} U_{s,t}\phi^{x,s}(y_u, u) \Big|_{u=t} = U_{t,t}^{-1}\phi^{\phi^{x,s}(y_t, t), t} \frac{d}{dt} U_{u,t}\phi^{\phi^{x,s}(y_u, u), u} \Big|_{u=t} \quad (2.5)$$

and therefore that U is a (two parameter) semigroup. We use (2.3) to economize the notation $U_{s,t}\phi^{x,s}(y, \cdot)$ to $U_{s,t}\phi^{x,s}$.

Now, define the \mathcal{F}_t -stopping time

$$\tau^{x,s} = \min(t_0^{x,s}, \inf\{t > s : Y_t^s \notin \mathcal{N}^{x,s} \text{ or } (\phi^{x,s}(Y_t^s, t), t) \notin D_T\})$$

and let

$$\mathcal{R}^{x,s} = \bigcup_{t \geq 0} \{(y, t) : P((Y_t^s, t) \in B_{(y,t)}(\delta), t \leq \tau^{x,s}) > 0 \quad \forall \delta > 0\}. \quad (2.6)$$

There is structure that can be imposed upon ϕ, U that will turn out to be equivalent to the existence of our explicit strong local solutions.

Definition 1. An (x, s, σ, h) -representation is a pair $\phi^{x,s}, U_{s,t}$ satisfying $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ such that the following system of differential equations:

$$\nabla_y \phi^{x,s}(y, t) = \sigma(\phi^{x,s}(y, t), t) U_{s,t}^{-1} \phi^{x,s}, \quad (2.7)$$

$$\partial_t \phi^{x,s}(y, t) = h(\phi^{x,s}(y, t), t) \quad (2.8)$$

hold for all $(y, t) \in \mathcal{R}^{x,s}$ and $\partial_s \nabla_y \phi^{x,s}(0, s)$, $\partial_s \partial_t \phi^{x,s}(0, s)$, $\partial_{x_i} \nabla_y \phi^{x,s}(0, s)$ and $\partial_{x_i} \partial_t \phi^{x,s}(0, s)$ exist as continuous functions. Here and below, $\partial_t \phi^{x,s}(0, s)$ means $\partial_t \phi^{x,s}(0, t) \Big|_{t=s}$.

Now, our explicit solutions are:

$$X_t^{x,s} = \phi(Y_t, t) = \phi^{x,s}(Y_t^s, t) \text{ on } [s, \tau^{x,s}). \quad (2.9)$$

Our first main result establishes two necessary and sufficient conditions for all $X^{x,s}$, defined in (2.9), to be strong local solutions to

$$dX_t = b(X_t, t)dt + \sigma(X_t, t)dW_t, \quad X_s = x \quad (2.10)$$

on $[s, \tau^{x,s})$. The function h is always related to b through (1.4) and $U_{s,t}\phi^{x,s}$ comes into the necessary and sufficient commutator conditions through generator

$$A(x, s) = \frac{d}{dt} U_{s,t}\phi^{x,s} \Big|_{t=s}. \quad (2.11)$$

It follows from (2.3) that A does not depend upon y .

Theorem 1. *The following are equivalent:*

- a) $\sigma \in C^1(D_T; \mathbb{R}^{p \times d})$, $h \in C^1(D_T; \mathbb{R}^p)$, *there is a unique strong solution to (2.10) on $[s, \tau^{x,s}]$ for each $(x, s) \in D_T$, and this solution has explicit form $\phi^{x,s}(Y_t^s, t)$ with $\phi^{x,s}, U_{s,t}$ satisfying $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$.*
- b) *There is a (x, s, σ, h) -representation $\phi^{x,s}, U_{s,t}$ for each $(x, s) \in D_T$.*
- c) $\sigma \in C^1(D_T; \mathbb{R}^{p \times d})$, $h \in C^1(D_T; \mathbb{R}^p)$ *and the following commutator conditions hold on D_T :*

$$(\nabla_\varphi \sigma_k) \sigma_j = (\nabla_\varphi \sigma_j) \sigma_k, \text{ for all } j, k \in \{1, \dots, d\}, \quad (2.12)$$

$$(\nabla_\varphi h) \sigma_j = (\nabla_\varphi \sigma_j) h + \partial_t \sigma_j - \sigma A_j, \text{ for all } 1 \leq j \leq d. \quad (2.13)$$

Remark 1. *Theorem 1 simplifies in the time-invariant h, σ coefficient case. Clearly, one only needs to check the commutator conditions on D versus D_T . However, the second commutator condition actually changes in form to:*

$$(\nabla_\varphi h) \sigma_j - (\nabla_\varphi \sigma_j) h = \sigma B_j, \text{ for all } 1 \leq j \leq d, \quad (2.14)$$

where $B(\varphi) = -A(\varphi, 0)$. Indeed, the left hand side of (2.14) does not depend on time so the right side can not either.

Remark 2. *Theorem 1 also simplifies when $d = 1$, which corresponds to appending a deterministic equation and allowing time dependence to the case considered in Kouritzin (2000). In this $d = 1$ case, (2.12) is automatically true and (2.13) becomes*

$$(\nabla_\varphi h) \sigma = (\nabla_\varphi \sigma) h + \partial_t \sigma - \sigma A. \quad (2.15)$$

Often, we are interested in establishing the representation for a given stochastic differential equation. In this case, the commutator conditions can be used quickly to determine if such a representation is possible. The easiest way to ensure (2.12) is to have each column a constant multiple of another $\sigma_j = c_j \sigma_1$ for all j say. However, there are other possibilities.

Example 1. *Let $p = d = 2$ and $D \subset \mathbb{R}$ be a domain. Suppose a, e, f, g, m, n are $C^2(D)$ -functions and our Fisk-Stratonovich equation has time-invariant coefficients:*

$$h(\varphi_1, \varphi_2) = \begin{pmatrix} f(\varphi_1) g(\varphi_2) \\ m(\varphi_1) n(\varphi_2) \end{pmatrix}, \quad \sigma(\varphi_1, \varphi_2) = \begin{pmatrix} a(\varphi_1) & 0 \\ e(\varphi_2) & e(\varphi_2) \end{pmatrix}. \quad (2.16)$$

Moreover, suppose $a(\varphi_1)$ and $e(\varphi_2)$ are never 0. Then, σ is always non-singular and it follows by (2.7) as well as the mean value theorem that for any $u \in [s, t]$

$$\phi^{x,s}(y, u) - \phi^{x,s}(\hat{y}, u) = \sigma(\phi^{x,s}(y^*, u)) U_{s,u}^{-1} \phi^{x,s} \cdot (y - \hat{y})$$

with $y^* \in \mathcal{N}^{x,s}$ for $y, \hat{y} \in \mathcal{N}^{x,s}$ and any possible representation $\phi^{x,s}, U_{s,t}$. Hence, $\phi^{x,s}(y, u) = \phi^{x,s}(\hat{y}, u) \leftrightarrow y = \hat{y}$. Therefore, it follows from (2.3) that $U_{s,u}$ can not depend upon $\phi^{x,s}(y, u)$ for any $u \in [s, t]$ and B in (2.14) is constant by (2.11). Now,

$$\nabla_\varphi h = \begin{pmatrix} f'(\varphi_1) g(\varphi_2) & f(\varphi_1) g'(\varphi_2) \\ m'(\varphi_1) n(\varphi_2) & m(\varphi_1) n'(\varphi_2) \end{pmatrix} \quad (2.17)$$

and

$$\nabla_{\varphi}\sigma_2 = \begin{pmatrix} 0 & 0 \\ 0 & e'(\varphi_2) \end{pmatrix}, \quad \nabla_{\varphi}\sigma_1 = \begin{pmatrix} a'(\varphi_1) & 0 \\ 0 & e'(\varphi_2) \end{pmatrix} \quad (2.18)$$

so the first commutator condition (2.12) is fine since

$$\nabla_{\varphi}\sigma_1\sigma_2 = \begin{pmatrix} 0 \\ e'(\varphi_2)e(\varphi_2) \end{pmatrix} = \nabla_{\varphi}\sigma_2\sigma_1. \quad (2.19)$$

Moreover,

$$\nabla_{\varphi}h\sigma_2 - \nabla_{\varphi}\sigma_2h = \begin{pmatrix} e(\varphi_2)f(\varphi_1)g'(\varphi_2) \\ m(\varphi_1)(e(\varphi_2)n'(\varphi_2) - e'(\varphi_2)n(\varphi_2)) \end{pmatrix} \quad (2.20)$$

and

$$\nabla_{\varphi}h\sigma_1 - \nabla_{\varphi}\sigma_1h = \begin{pmatrix} af'g + efg' - a'fg \\ am'n + emn' - e'mn \end{pmatrix}. \quad (2.21)$$

On the other hand, denoting $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, we have

$$\sigma B = \begin{pmatrix} ab_{11} & ab_{12} \\ eb_{11} + eb_{21} & eb_{12} + eb_{22} \end{pmatrix}. \quad (2.22)$$

Hence, by (2.14) there is an explicit solution if and only if

$$\begin{pmatrix} af'g + efg' - a'fg & efg' \\ am'n + emn' - e'mn & m(en' - e'n) \end{pmatrix} = \begin{pmatrix} ab_{11} & ab_{12} \\ eb_{11} + eb_{21} & eb_{12} + eb_{22} \end{pmatrix} \quad (2.23)$$

for constants $b_{11}, b_{12}, b_{21}, b_{22}$. If $f = c_1a$, $n = c_2e$, $eg' = c_3$ and $m'a = c_4$ for some constants c_1, c_2, c_3, c_4 , then it is easy to show that this condition is met with $b_{22} = -c_1c_3$, $b_{21} = c_2c_4 - c_1c_3$ and $b_{11} = b_{12} = c_1c_3$ so the representation holds for

$$h(\varphi_1, \varphi_2) = \begin{pmatrix} \alpha \frac{g(\varphi_2)}{m'(\varphi_1)} \\ \beta \frac{m(\varphi_1)}{g'(\varphi_2)} \end{pmatrix}, \quad \sigma(\varphi_1, \varphi_2) = \begin{pmatrix} \frac{\gamma}{m'(\varphi_1)} & 0 \\ \frac{\delta}{g'(\varphi_2)} & \frac{\delta}{g'(\varphi_2)} \end{pmatrix}, \quad (2.24)$$

where $\alpha = c_1c_4$, $\beta = c_2c_3$, $\gamma = c_4$, $\delta = c_3$ are any constants and g, m are C^2 -functions with $\frac{1}{m'(\varphi_1)}, \frac{1}{g'(\varphi_1)} \in C^1(D)$.

Example 2. In a similar manner, it follows that

$$h(\varphi_1, \varphi_2) = \begin{pmatrix} \alpha \frac{g(\varphi_2)}{m'(\varphi_1)} \\ \beta \frac{m(\varphi_1)}{g'(\varphi_2)} \end{pmatrix}, \quad \sigma(\varphi_1, \varphi_2) = \begin{pmatrix} \frac{\gamma}{m'(\varphi_1)} & 0 \\ 0 & \frac{\delta}{g'(\varphi_2)} \end{pmatrix}, \quad (2.25)$$

for any constants $\alpha, \beta, \gamma, \delta$, also has a representation.

There was significant work done in the previous examples and we still did not have the representation functions. The next example is the key to solving for complete representations and will be used in the following section.

Example 3. Suppose $\sigma(\varphi, t) = \begin{pmatrix} I_r & \bar{\kappa}(\varphi, t) \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{p \times d}$ satisfies (2.12). We will find the possible h, b satisfying (2.13) and the corresponding representations $U_{s,t}, \phi^{x,s}$ by Theorem 1.

Notation: As always, φ is a variable and ϕ is the representation function. Further, let $\bar{x} = (x_1, \dots, x_r)$, $\tilde{x} = (x_{r+1}, \dots, x_d)$, $\bar{\varphi} = (\varphi_1, \dots, \varphi_r)$, $\tilde{\varphi} = (\varphi_{r+1}, \dots, \varphi_d)$, $\tilde{D} = \{\tilde{\varphi} : (\bar{\varphi}, \tilde{\varphi}) \in D \text{ for some } \bar{\varphi}\}$, $\tilde{D}_T = \tilde{D} \times [0, T)$,

$$\phi^{x,s}(y, t) = \begin{pmatrix} \bar{\phi}^{x,s}(y, t) \\ \tilde{\phi}^{x,s}(y, t) \end{pmatrix}, \quad h = \begin{pmatrix} \bar{h} \\ \tilde{h} \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad (2.26)$$

where $A_{11} \in \mathbb{R}^{r \times r}$. Finally, we let

$$\bar{\beta}(\varphi, t) = -A_{11}(\varphi, t) - \bar{\kappa}(\varphi, t)A_{21}(\varphi, t), \quad (2.27)$$

which will appear often below.

Step 1: Interpret (2.7) and \mathcal{C}_2 condition (2.3) on $U_{s,t}, A$.

Suppose $u \in [s, t]$. By (2.7) as well as the mean value theorem

$$\begin{pmatrix} \bar{\phi}^{x,s}(y, u) - \bar{\phi}^{x,s}(\hat{y}, u) \\ \tilde{\phi}^{x,s}(y, u) - \tilde{\phi}^{x,s}(\hat{y}, u) \end{pmatrix} = \begin{pmatrix} I_r & \bar{\kappa}(\phi^{x,s}(y^*, u), u) \\ 0 & 0 \end{pmatrix} U_{s,u}^{-1} \phi^{x,s} \cdot (y - \hat{y}) \quad (2.28)$$

with $y^* \in \mathcal{N}^{x,s}$ for $y, \hat{y} \in \mathcal{N}^{x,s}$ and any possible representation $\phi^{x,s}$. Hence, $\bar{\phi}^{x,s}(y, u) \neq \bar{\phi}^{x,s}(\hat{y}, u)$ implies $y \neq \hat{y}$. Therefore, it follows from (2.3) that $U_{s,t} \phi^{x,s}$ can not depend upon $\bar{\phi}^{x,s}(y, u)$ for any $u \in [s, t]$, which implies $U_{s,t} \phi \doteq U_{s,t} \tilde{\phi}$ only depends on $\tilde{\phi}, t$. This also means by (2.11) that

$$A(\varphi, t) = \frac{d}{dt} U_{u,t} \tilde{\phi}^{\varphi, u} \Big|_{u=t}. \quad (2.29)$$

Step 2: Interpret commutator conditions on $\bar{\kappa}, h$.

Let e_i denote the i^{th} column of I_p so $\sigma_i = e_i$ for $i \leq r$. We have by (2.12), that

$$\begin{pmatrix} \nabla_{\varphi} \bar{\kappa}_{j-r} \\ 0 \end{pmatrix} e_i = 0 \quad \forall i \in \{1, 2, \dots, r\}, j \in r+1, \dots, d, \quad (2.30)$$

which establishes that $\bar{\kappa}(\tilde{\varphi}, t)$ can only depend upon $\tilde{\varphi}, t$. This is the only restriction on $\bar{\kappa}$ from (2.12). By (2.13), we find

$$\nabla_{\varphi} \begin{pmatrix} \bar{h} \\ \tilde{h} \end{pmatrix} \sigma_j - \nabla_{\varphi} \sigma_j \begin{pmatrix} \bar{h} \\ \tilde{h} \end{pmatrix} = \begin{pmatrix} \bar{\beta} & \partial_t \bar{\kappa} - A_{12} - \bar{\kappa} A_{22} \\ 0 & 0 \end{pmatrix}_j \quad (2.31)$$

so $\nabla_{\tilde{\varphi}} \tilde{h} = 0$, implying $\tilde{h}(\varphi) \in C^1(\tilde{D}_T, \mathbb{R}^{p-r})$ only depends upon $\tilde{\varphi}, t$, and

$$\nabla_{\tilde{\varphi}} \bar{h} = \bar{\beta}, \quad (2.32)$$

$$\bar{\beta} \bar{\kappa} = [(\nabla_{\tilde{\varphi}} \bar{\kappa}_1) \tilde{h}, \dots, (\nabla_{\tilde{\varphi}} \bar{\kappa}_{d-r}) \tilde{h}] + \partial_t \bar{\kappa} - A_{12} - \bar{\kappa} A_{22}. \quad (2.33)$$

Hence, it follows from (2.7,2.8,2.2) that $\tilde{\phi}^{x,s}$ satisfies

$$\nabla_y \tilde{\phi}^{x,s}(y, t) = 0, \quad (2.34)$$

$$\partial_t \tilde{\phi}^{x,s}(y, t) = \tilde{h}(\tilde{\phi}^{x,s}(y, t), t), \quad (2.35)$$

$$\tilde{\phi}^{x,s}(0, s) = \tilde{x}, \quad (2.36)$$

which implies that $\tilde{\phi}$ does not depend upon $\bar{\phi}$ nor y . Moreover, by (2.29) and (2.27), we conclude that $A(\varphi, t) \doteq A(\tilde{\varphi}, t)$ and $\bar{\beta}(\varphi, t) \doteq \bar{\beta}(\tilde{\varphi}, t)$ only depend on $\tilde{\varphi}, t$.

Step 3: Determine possible h, b .

By (2.32), we find

$$\bar{h}(\bar{\varphi}, \tilde{\varphi}, t) = \bar{\beta}(\tilde{\varphi}, t)\bar{\varphi} + \bar{\theta}(\tilde{\varphi}, t) \quad (2.37)$$

for some C^1 -function $\bar{\theta}$. Hence, the possible $h(\bar{\varphi}, \tilde{\varphi}, t) = \begin{pmatrix} \bar{h}(\bar{\varphi}, \tilde{\varphi}, t) \\ \tilde{h}(\tilde{\varphi}, t) \end{pmatrix}$ are:

$$\begin{aligned} \tilde{h} &\in C^1(\tilde{D}_T, \mathbb{R}^{p-r}), \\ \bar{h} &\in \left\{ \bar{\theta}(\tilde{\varphi}, t) + \bar{\beta}(\tilde{\varphi}, t)\bar{\varphi} : \bar{\beta} \in C^1(\tilde{D}_T, \mathbb{R}^{r \times r}); \bar{\theta} \in C^1(\tilde{D}_T, \mathbb{R}^r) \right\}. \end{aligned} \quad (2.38)$$

From (1.4) and fact $\bar{\kappa}(\tilde{\varphi}, t)$ only depends on $\tilde{\varphi}, t$, we find that

$$b = h + \frac{1}{2} \sum_{j=1}^d \{ \nabla_{\varphi} \sigma_j \} \sigma_j = h. \quad (2.39)$$

Free Parameters: $A_{21}, A_{22}, \bar{\kappa}, \bar{\beta}, \bar{\theta}$ and \tilde{h} can be anything (subject to dimensionality and dependency on only $\tilde{\varphi}, t$). A_{12} is then determined by (2.33) and A_{11} by (2.27). $\bar{\beta}$ and $\bar{\theta}$ also determine the possible \bar{h} above and $\phi^{x,s}$ below. Different choices of $\bar{\kappa}, \bar{\beta}, \bar{\theta}$ and \tilde{h} will result in different solutions. However, there is no loss in generality in taking A_{21}, A_{22} to be zero.

Step 4: Interpret differential system for $\phi^{x,s}$.

Since $\phi^{x,s} = \begin{pmatrix} \bar{\phi} \\ \tilde{\phi} \end{pmatrix}$ satisfies (2.8,2.2), $\tilde{\phi}$ must be of the form

$$\partial_t \tilde{\phi} = \tilde{h}(\tilde{\phi}, t), \quad \text{s.t. } \tilde{\phi}(s) = \tilde{x}. \quad (2.40)$$

We let \tilde{X}_t denote the solution of this differential equation. Next, since ϕ satisfies (2.7), $\bar{\phi}$ must be of the form

$$\bar{\phi}^{x,s}(y, t) = \bar{c}(t) + \begin{bmatrix} I_r & \bar{\kappa}(\tilde{X}_t, t) \end{bmatrix} U_{s,t}^{-1} \tilde{\phi}^{x,s} y, \quad (2.41)$$

for some $\bar{c} \in C^1([0, T]; \mathbb{R}^r)$. Differentiating in t , noting by (2.29) (with $\tilde{\varphi} = \tilde{X}_t$) that

$$A(\tilde{X}_t, t) = \frac{d}{dt} U_{u,t} \tilde{\phi}^{\tilde{X}_t, u} \Big|_{u=t}, \quad (2.42)$$

and using (2.41, 2.42, 2.4, 2.40, 2.33, 2.27), one has (with $U_{s,t}^{-1} = U_{s,t}^{-1} \tilde{\phi}^{x,s}$) that

$$\begin{aligned} & \partial_t \bar{\phi}(y, t) \\ &= \bar{c}'(t) - \begin{bmatrix} I & \bar{\kappa}(\tilde{X}_t, t) \end{bmatrix} A(\tilde{X}_t, t) U_{s,t}^{-1} y \\ &+ \begin{bmatrix} 0 & \partial_t \bar{\kappa}(\tilde{X}_t, t) + \nabla_{\tilde{\varphi}} \bar{\kappa}_1(\tilde{X}_t, t) \tilde{h}(\tilde{X}_t, t), \dots, \nabla_{\tilde{\varphi}} \bar{\kappa}_{d-r}(\tilde{X}_t, t) \tilde{h}(\tilde{X}_t, t) \end{bmatrix} U_{s,t}^{-1} y \\ &= \bar{c}'(t) + \bar{\beta}(\tilde{X}_t, t) [I \quad \bar{\kappa}(\tilde{X}_t, t)] U_{s,t}^{-1} y \\ &= \bar{c}'(t) + \bar{\beta}(\tilde{X}_t, t) (\bar{\phi}(y, t) - \bar{c}(t)). \end{aligned} \quad (2.43)$$

On the other hand, by (2.8) and (2.37)

$$\partial_t \bar{\phi}(y, t) = \bar{\theta}(\tilde{X}_t, t) + \bar{\beta}(\tilde{X}_t, t) \bar{\phi}(y, t). \quad (2.44)$$

Comparing (2.43) and (2.44), one has that

$$\bar{c}'(t) = \bar{\theta}(\tilde{X}_t, t) + \bar{\beta}(\tilde{X}_t, t) \bar{c}(t) \quad \text{subject to } \bar{c}(s) = \bar{x}. \quad (2.45)$$

Step 5: Determine U in terms of $\bar{\kappa}$, $\bar{\beta}$ and θ .

We just need A to satisfy (2.27, 2.33) so there is no loss of generality in taking

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} (\tilde{\varphi}, t) = \begin{pmatrix} -\bar{\beta} & [(\nabla_{\tilde{\varphi}} \bar{\kappa}_1) \tilde{h}, \dots, (\nabla_{\tilde{\varphi}} \bar{\kappa}_{d-r}) \tilde{h}] + \partial_t \bar{\kappa} - \bar{\beta} \bar{\kappa} \\ 0 & 0 \end{pmatrix} (\tilde{\varphi}, t). \quad (2.46)$$

By (2.42), (2.4) and (2.46), we know

$$\begin{aligned} \partial_t U_{s,t} \tilde{X} &= (U_{s,t} \tilde{X}) A(\tilde{X}_t, t) \\ &= U_{s,t} \tilde{X} \begin{pmatrix} -\bar{\beta} & \{[(\nabla_{\tilde{\varphi}} \bar{\kappa}_1) \tilde{h}, \dots, (\nabla_{\tilde{\varphi}} \bar{\kappa}_{d-r}) \tilde{h}] + \partial_t \bar{\kappa} - \bar{\beta} \bar{\kappa}\} \\ 0 & 0 \end{pmatrix} (\tilde{X}_t, t) \end{aligned} \quad (2.47)$$

subject to $U_{s,s} \tilde{X} = U_{s,s} \tilde{x} = I_d$. Now, suppose that $T_{u,t}$ is the two parameter semigroup:

$$\frac{d}{dt} T_{u,t} = -T_{u,t} \bar{\beta}(\tilde{X}_t, t), \quad \forall t \geq u \quad \text{subject to } T_{u,u} = I_r. \quad (2.48)$$

Then, the solution of (2.47) is

$$U_{s,t} \tilde{X} = \begin{pmatrix} T_{s,t} & T_{s,t} \bar{\kappa}(\tilde{X}_t, t) - \bar{\kappa}(\tilde{X}_s, s) \\ 0 & I_{d-r} \end{pmatrix}, \quad (2.49)$$

and so

$$U_{s,t}^{-1} \tilde{X} = \begin{pmatrix} T_{s,t}^{-1} & T_{s,t}^{-1} \bar{\kappa}(\tilde{X}_s, s) - \bar{\kappa}(\tilde{X}_t, t) \\ 0 & I_{d-r} \end{pmatrix}. \quad (2.50)$$

Moreover, it follows by (2.45) that \bar{c} can also be expressed in terms of $T_{s,t}^{-1}$.

Step 6: Solution Algorithm.

a: Check $\bar{\kappa}$ only depends upon $\tilde{\varphi}, t$. This must be true by Step 2.

b: Choose any functions $\bar{\beta} \in C^1(\tilde{D}_T, \mathbb{R}^{r \times r}); \bar{\theta} \in C^1(\tilde{D}_T, \mathbb{R}^r)$ and $\tilde{h} \in C^1(\tilde{D}_T, \mathbb{R}^{p-r})$ for drift of the form $b(\bar{\varphi}, \tilde{\varphi}, t) = h(\bar{\varphi}, \tilde{\varphi}, t) = \begin{pmatrix} \bar{\theta}(\tilde{\varphi}, t) + \bar{\beta}(\tilde{\varphi}, t)\bar{\varphi} \\ \tilde{h}(\tilde{\varphi}, t) \end{pmatrix}$.

These are the only possible drifts by Step 3.

c: Solve

$$\tilde{X}'_t = \tilde{h}(\tilde{X}_t, t) \quad \text{subject to } \tilde{X}_s = \tilde{x}$$

d: Solve

$$\frac{d}{dt}T_{s,t} = -T_{s,t}\bar{\beta}(\tilde{X}_t, t), \quad \forall t \geq s \quad \text{subject to } T_{s,s} = I_r. \quad (2.51)$$

Then, set

$$U_{s,t}\tilde{X} = \begin{pmatrix} T_{s,t} & T_{s,t}\bar{\kappa}(\tilde{X}_t, t) - \bar{\kappa}(\tilde{X}_s, s) \\ 0 & I_{d-r} \end{pmatrix}, \quad (2.52)$$

$$U_{s,t}^{-1}\tilde{X} = \begin{pmatrix} T_{s,t}^{-1} & T_{s,t}^{-1}\bar{\kappa}(\tilde{X}_s, s) - \bar{\kappa}(\tilde{X}_t, t) \\ 0 & I_{d-r} \end{pmatrix}, \quad (2.53)$$

$$\bar{c}(t) = T_{s,t}^{-1}\bar{x} + T_{s,t}^{-1} \int_s^t T_{s,u}\bar{\theta}(\tilde{X}_u, u)du. \quad (2.54)$$

e: Divide $\phi = \begin{pmatrix} \bar{\phi} \\ \tilde{\phi} \end{pmatrix}$ and set $\tilde{\phi}(t) = \tilde{X}_t$,

$$\bar{\phi}(y, t) = \bar{c}(t) + \begin{bmatrix} I_r & \bar{\kappa}(\tilde{X}_t, t) \end{bmatrix} (U_{s,t}^{-1}\tilde{X})y.$$

The preceding example was intuitively pleasing: We showed you could indeed represent *linear* SDEs using a single Gaussian stochastic integral. Further, we showed that we could *append* an ordinary differential equation ($d\tilde{X}_t = \tilde{h}(\tilde{X}_t)dt$) and use its solution within the coefficients of the stochastic differential equation. Finally, we showed how to construct the solution. While none of this is surprising, it does explain our necessary and sufficient conditions. In the next section, we will show how to combine this example with diffeomorphisms to handle the general case with nonlinear coefficients.

3. CONSTRUCTION RESULTS AND EXAMPLES

When one explicit solution exists, there will be a whole class of such solutions corresponding to distinct b 's. We now identify the b 's, ϕ 's and U 's for these solutions corresponding to a given σ . This is done by using local diffeomorphisms to convert the general case to the case of Example 3. The idea is based upon the following simple lemma.

Lemma 1. *Suppose $D \subset \mathbb{R}^p$ is a domain, $T > 0$, $D_T = D \times [0, T)$, $\hat{\Lambda} \doteq \begin{pmatrix} \Lambda_t \\ t \end{pmatrix} : D_T \rightarrow \hat{\Lambda}(D_T) \subset \mathbb{R}^{p+1}$ is a C^2 -diffeomorphism and $\sigma, b, h, \{\phi^{x,s}\}_{(x,s) \in D_T}$,*

$\{U_{s,t}\phi^{x,s}\}_{(x,s)\in D_T, s\leq t<T}$, A satisfy Conditions $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ as well as equations (1.4, 2.11). Let $\widehat{D}_T = \widehat{\Lambda}(D_T)$,

$$\begin{aligned}\widehat{\sigma} &= \{(\nabla_\varphi \Lambda_t)\sigma\} \circ \widehat{\Lambda}^{-1}, \quad \widehat{h} = \{(\nabla_\varphi \Lambda_t)h\} \circ \widehat{\Lambda}^{-1}, \\ \widehat{b} &= \left\{ (\nabla_\varphi \Lambda_t)b + \frac{1}{2} \sum_{j=1}^d \sum_{i,k=1}^p (\partial_{\varphi_i} \partial_{\varphi_k} \Lambda_t) \sigma_{i,j} \sigma_{k,j} \right\} \circ \widehat{\Lambda}^{-1}, \\ \widehat{\phi}^{x,s}(y, t) &= \Lambda_t \circ \phi^{\widehat{\Lambda}^{-1}(x,s)}(y, t), \\ \widehat{U}_{s,t} \widehat{\phi}^{x,s} &= U_{s,t} \phi^{\widehat{\Lambda}^{-1}(x,s)}, \\ \widehat{A} &= A \circ \widehat{\Lambda}^{-1}.\end{aligned}$$

Then, $\widehat{\sigma}, \widehat{b}, \widehat{h}, \{\widehat{\phi}^{x,s}\}_{(x,s)\in \widehat{D}_T}, \widehat{U}, \widehat{A}$ satisfy Conditions $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ as well as equations (1.4, 2.11) on \widehat{D}_T . Moreover,

- i) $\widehat{\phi}, \widehat{U}$ is a $(x, s, \widehat{\sigma}, \widehat{h})$ -representation for each $(x, s) \in \widehat{D}_T$ if and only if ϕ, U is a (x, s, σ, h) -representation for each $(x, s) \in D_T$.
- ii) (2.12) holds if and only

$$(\nabla_\varphi \widehat{\sigma}_k) \widehat{\sigma}_j = (\nabla_\varphi \widehat{\sigma}_j) \widehat{\sigma}_k, \text{ on } \widehat{D}_T \text{ for all } j, k \in \{1, \dots, d\}. \quad (3.1)$$

- iii) (2.13) holds if and only

$$(\nabla_\varphi \widehat{h}) \widehat{\sigma}_j = (\nabla_\varphi \widehat{\sigma}_j) \widehat{h} + \partial_t \widehat{\sigma}_j - \widehat{\sigma} \widehat{A}_j, \text{ on } \widehat{D}_T \text{ for all } 1 \leq j \leq d. \quad (3.2)$$

Remark 3. In the time-homogeneous case, we can deal with B instead of A and set $\widehat{B} = B \circ \Lambda_0^{-1}$.

Proof. This lemma follows by direct calculation. Perhaps, the fastest way to verify the commutator conditions is to think of (1.3) as a time-homogeneous equation

$$d \begin{bmatrix} X_t \\ t \end{bmatrix} = \begin{bmatrix} h(X_t, t) \\ 1 \end{bmatrix} dt + \begin{bmatrix} \sigma(X_t, t) \\ 0 \end{bmatrix} \bullet dW_t, \quad \begin{bmatrix} X_s \\ s \end{bmatrix} = \begin{bmatrix} x \\ s \end{bmatrix}$$

on $[s, \tau^{x,s})$, by appending the trivial equation $t = t$ and thinking of t as an additional state variable. Then, verifying (2.13) is equivalent to (3.2) is the same as verifying

$$\begin{aligned} \left(\nabla \begin{bmatrix} h \\ 1 \end{bmatrix} \right) \begin{bmatrix} \sigma_j \\ 0 \end{bmatrix} &= \left(\nabla \begin{bmatrix} \sigma_j \\ 0 \end{bmatrix} \right) \begin{bmatrix} h \\ 1 \end{bmatrix} - \begin{bmatrix} \sigma \\ 0 \end{bmatrix} A_j \\ \Leftrightarrow \left(\nabla \begin{bmatrix} \widehat{h} \\ 1 \end{bmatrix} \right) \begin{bmatrix} \widehat{\sigma}_j \\ 0 \end{bmatrix} &= \left(\nabla \begin{bmatrix} \widehat{\sigma}_j \\ 0 \end{bmatrix} \right) \begin{bmatrix} \widehat{h} \\ 1 \end{bmatrix} - \begin{bmatrix} \widehat{\sigma} \\ 0 \end{bmatrix} \widehat{A}_j, \end{aligned}$$

which avoids $\partial_t \sigma_j$ and Λ_t if we express $(\widehat{h}^T, 1)^T$ and $(\widehat{\sigma}_j^T, 0)^T$ in terms of $\widehat{\Lambda}$. \square

The idea behind this lemma is that with some diffeomorphism $\widehat{\sigma} = \begin{pmatrix} I_r & \overline{\kappa} \\ 0 & 0 \end{pmatrix}$ so we can use Example 3 to solve for the possible \widehat{h} and the representations

$\widehat{\phi}^{x,s}, \widehat{U}^{x,s}$. Unfortunately, it is sometimes impossible to have a single diffeomorphism for all of D_T and, even when it is possible, we may not know that until after local diffeomorphisms are constructed and one of them is extendable to all of D_T .

Definition 2. Suppose $(x, s) \in D_T$. Then, an (x, s) -local diffeomorphism $(O^{x,s}, \widehat{\Lambda}^{x,s})$ is a bijection $\widehat{\Lambda}^{x,s} : O^{x,s} \rightarrow \widehat{\Lambda}^{x,s}(O^{x,s})$ such that $\widehat{\Lambda}^{x,s} \in C^2(O^{x,s}; \mathbb{R}^{p+1})$, where $O^{x,s} \subset D_T$ is a (relatively open) neighbourhood of x, s . We define $\nabla \widehat{\Lambda}^{-1}(\widehat{\Lambda}(\varphi, t))$ to be $\left[\nabla \widehat{\Lambda}(\varphi, t) \right]^{-1}$ for $(\varphi, t) \in O^{x,s}$.

We imposed sufficient differentiability on our local diffeomorphisms for our uses to follow. Our (x, s) -local diffeomorphisms will take the form $\widehat{\Lambda} = \begin{pmatrix} \Lambda_t \\ t \end{pmatrix}$ with Λ_t being constructed from σ under the conditions:

- D : Let $D \subset \mathbb{R}^p$ be a bounded convex domain, $T > 0$ and $D_T = D \times [0, T)$.
- ∂_1 : $\sigma \in C^1(D_T; \mathbb{R}^{p \times d})$.
- H_r : The rank of σ is r on D_T with the first r rows having full row rank.
- B : $(\nabla_\varphi \sigma_j) \sigma_k - (\nabla_\varphi \sigma_k) \sigma_j = 0$ on D_T , for $1 \leq j, k \leq d$ and $(x, s) \in D_T$.

To ensure the row rank part of H_r , we can just permute the rows of (1.2), amounting to relabeling the $\{X_t^i\}_{i=1}^p$.

Proposition 1. Suppose $[D, \partial_1, H_r, B]$ hold. Then, there exists an (x, s) -local diffeomorphism $(O^{x,s}, \widehat{\Lambda}^{x,s})$ and a constant permutation matrix π such that

$$\widehat{\sigma} \doteq \{(\nabla_\varphi \Lambda_t) \sigma \pi\} \circ \widehat{\Lambda}^{-1} = \begin{pmatrix} I_r & \overline{\kappa} \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{p \times d} \text{ on } \widehat{\Lambda}(O^{x,s}),$$

where $\overline{\kappa} \in C^1(\widehat{\Lambda}(O^{x,s}); \mathbb{R}^{r \times (d-r)})$ does not depend on $\varphi_1, \dots, \varphi_r$.

Proof. Provided in the Appendix. □

Remark 4. The permutation matrix π permutes the columns of σ . We label the permuted diffusion coefficient $\sigma^\pi = \sigma \pi$ and note that

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t = b(X_t)dt + \sigma^\pi(X_t)dW_t^\pi,$$

where $W^\pi = \pi^{-1}W$ is a permutation of the Brownian motions W . Also, the Stratonovich drift h remains the same by (1.4).

Remark 5. It follows from the proof in the Appendix that the diffeomorphism can have the form $\widehat{\Lambda} = \widehat{\Lambda}_r \circ \dots \circ \widehat{\Lambda}_2 \circ \widehat{\Lambda}_1$ for any diffeomorphisms $\widehat{\Lambda}_i : \widehat{\Lambda}_{i-1} \circ \dots \circ \widehat{\Lambda}_2 \circ \widehat{\Lambda}_1(D_T) \rightarrow \mathbb{R}^{p+1}$ satisfying $\{\nabla \widehat{\Lambda}_i \dots \nabla \widehat{\Lambda}_2 \nabla \widehat{\Lambda}_1 \sigma_i^\pi\} \circ \widehat{\Lambda}_1^{-1} \circ \widehat{\Lambda}_2^{-1} \circ \dots \circ \widehat{\Lambda}_i^{-1} = e_i$, where $(e_1 \ e_2 \ \dots \ e_p \ e_{p+1}) = I_{p+1}$ is the identity matrix. However, as will be seen below in Remark 7, this does not uniquely define the diffeomorphism.

Proposition 1 immediately provides us our second main theorem.

Theorem 2. Suppose $[D, \partial_1, H_r, B]$ hold, $h \in C^1(D_T; \mathbb{R}^p)$, $(x, s) \in D_T$ and W is an \mathbb{R}^d -valued standard Brownian motion. Then, there exists a stopping time $\tau > s$, a permutation matrix π and an (x, s) -local diffeomorphism $(O^{x,s}, \hat{\Lambda}^{x,s})$, as in Proposition 1 and Remark 5, such that

$$i) \quad \hat{\sigma} \doteq \{(\nabla_\varphi \Lambda_t) \sigma^\pi\} \circ \hat{\Lambda}^{-1} = \begin{pmatrix} I_r & \bar{\kappa} \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{p \times d} \text{ on } \hat{\Lambda}(O^{x,s}),$$

with $\bar{\kappa} \in C^1(\Lambda(O^{x,s}); \mathbb{R}^{r \times (d-r)})$ not depending on $\varphi_1, \dots, \varphi_r$ and ii) the Stratonovich SDE $dX_t = h(X_t)dt + \sigma(X_t) \bullet dW_t$, $X_s = x$ has a solution $X_t = \Lambda_t^{-1} \begin{pmatrix} \bar{X}_t \\ \tilde{X}_t \end{pmatrix}$ on $[0, \tau]$ if and only if the simpler SDE

$$d \begin{bmatrix} \bar{X}_t \\ \tilde{X}_t \end{bmatrix} = \hat{h} \begin{pmatrix} \bar{X}_t \\ \tilde{X}_t \end{pmatrix} dt + \begin{pmatrix} I_r & \bar{\kappa} \\ 0 & 0 \end{pmatrix} dW_t^\pi, \quad \begin{bmatrix} \bar{X}_s \\ \tilde{X}_s \end{bmatrix} = \Lambda_s(x)$$

has a solution on $[0, \tau]$, where $\hat{h} = (\nabla_\varphi \Lambda_t h + \partial_t \Lambda_t) \circ \hat{\Lambda}^{-1}$.

We stated the simpler SDE in terms of Itô integration. However, it follows by (1.4) and the nature of $\bar{\kappa}$ that this equation would have exactly the same form in terms of Stratonovich integration.

In this theorem we do not have a commutator condition for h so we can not guarantee the simple form of \hat{h} as in Example 3. This means that \tilde{X} is not in general deterministic nor is \bar{X} necessarily Gaussian. We also impose slightly stronger conditions on σ compared to Theorem 1 but gain information about the representation as local diffeomorphisms.

Kouritzin (2016) solves for a local diffeomorphism $\hat{\Lambda}$ of the form stated in Remark 5 corresponding to the (extended) Heston model, shows that it exists globally, finds the corresponding \hat{h} , and solves the SDEs. The use of the extended model means that our explicit Heston SDE solutions are weak not strong because the real Heston model corresponds to just part of the extended model that includes *extra randomness*. Also, this approach only works when a condition is imposed on the Heston parameters. When this condition is not true, one can still obtain an explicit weak solution by using Likelihoods and Girsanov's theorem to convert to the case where the condition is true.

For our final main result, we add back the commutator condition for h , and characterize all the solutions $X_t^{x,s} = \phi^{x,s}(Y_t, t)$ to (2.10) via Example 3. We do this through our basic set of parameters for (x, s) :

Definition 3. Let $\mathcal{P} = \mathcal{P}_\sigma^{x,s}$ be the set of all $(\hat{\Lambda}, \bar{\kappa}, \bar{\beta}, \bar{\theta}, \tilde{h}, \pi)$ such that

P0) π is a constant permutation matrix.

P1) $(O^{x,s}, \hat{\Lambda}^{x,s})$ is a (x, s) -local diffeomorphism, where $\hat{\Lambda}(\varphi, t) = \begin{bmatrix} \Lambda_t(\varphi) \\ t \end{bmatrix}$.

For convenience, we let $\Lambda_t = \begin{bmatrix} \bar{\Lambda}_t \\ \tilde{\Lambda}_t \end{bmatrix}$ with $\bar{\Lambda}_t \in \mathbb{R}^r$;

P2) $\bar{\kappa} \in C^1(\hat{\Lambda}(O); \mathbb{R}^{r \times (d-r)})$ depends only on $\varphi_{r+1}, \dots, \varphi_p$, and t ;

- P3) $\{(\nabla_\varphi \Lambda_t) \sigma^\pi\} \circ (\hat{\Lambda})^{-1} = \begin{pmatrix} I_r & \bar{\kappa} \\ 0 & 0 \end{pmatrix}$ on $\hat{\Lambda}(O)$;
P4) $\bar{\beta} \in C^1(\hat{\Lambda}(O); \mathbb{R}^{r \times r})$ depends only on $\varphi_{r+1}, \dots, \varphi_p$, and t ;
P5) $\bar{\theta} \in C^1(\hat{\Lambda}(O); \mathbb{R}^r)$ depends only on $\varphi_{r+1}, \dots, \varphi_p, t$;
P6) $\tilde{h} \in C^1(\hat{\Lambda}(O); \mathbb{R}^{p-r})$ depends only on $\varphi_{r+1}, \dots, \varphi_p, t$.

To each $(\hat{\Lambda}, \bar{\kappa}, \bar{\beta}, \bar{\theta}, \tilde{h}, \pi) \in \mathcal{P}$, we associate the following functions:

$$\begin{cases} \tilde{X} = \tilde{X}^{x,s} \in \mathbb{R}^{p-r} \text{ uniquely solves } \frac{d}{dt} \tilde{X}_t = \tilde{h}(\tilde{X}_t, t), \tilde{X}_s = \tilde{\Lambda}_s(x); \\ G(t) = \begin{pmatrix} I_r & \bar{\kappa}(\tilde{X}_t, t) \end{pmatrix} \in \mathbb{R}^{r \times d}; \\ \frac{d}{du} T_{s,u} = -T_{s,u} \bar{\beta}(\tilde{X}_u, u), \quad \forall u \geq s \text{ subject to } T_{s,s} = I_r; \\ U_{s,u} \tilde{X} = \begin{pmatrix} T_{s,u} & T_{s,u} \bar{\kappa}(\tilde{X}_u, u) - \bar{\kappa}(\tilde{X}_s, s) \\ 0 & I_{d-r} \end{pmatrix}; \\ U_{s,u}^{-1} \tilde{X} = \begin{pmatrix} T_{s,u}^{-1} & T_{s,u}^{-1} \bar{\kappa}(\tilde{X}_s, s) - \bar{\kappa}(\tilde{X}_u, u) \\ 0 & I_{d-r} \end{pmatrix}; \\ \bar{c}_s(t) = T_{s,t}^{-1} \bar{\Lambda}_s(x) + T_{s,t}^{-1} \int_s^t T_{s,u} \bar{\theta}(\tilde{X}_u, u) du. \end{cases} \quad (3.3)$$

The following theorem follows from Theorem 2, Theorem 1 (so the explicit solution implies B above) and Example 3. In particular, we must have

$$(\nabla_\varphi \Lambda_t h + \partial_t \Lambda_t) \circ \hat{\Lambda}^{-1} = \begin{pmatrix} \bar{h}(\bar{\varphi}, \bar{\varphi}, t) \\ \tilde{h}(\bar{\varphi}, t) \end{pmatrix} = \begin{pmatrix} \bar{\theta}(\bar{\varphi}, t) + \bar{\beta}(\bar{\varphi}, t) \bar{\varphi} \\ \tilde{h}(\bar{\varphi}, t) \end{pmatrix}, \quad (3.4)$$

which gives our possible drifts h in the following theorem.

Theorem 3. Suppose $[D, \partial_1, H_r]$ hold, $(x, s) \in D_T$ and $X_t^{x,s} = \phi^{x,s} \left(\int_s^t U_{s,u} \phi^{x,s} dW_u^\pi, t \right)$, with ϕ, U satisfying $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, solves (2.10) up to some stopping time $\tau^{x,s} > s$. Then, there exists $((O^{x,s}, \hat{\Lambda}^{x,s}), \bar{\kappa}, \bar{\beta}, \bar{\theta}, \tilde{h}, \pi) \in \mathcal{P}_\sigma^{x,s}$, and related functions \tilde{X}, G, U, \bar{c} defined by (3.3), such that

$$h = [\nabla_\varphi \Lambda_t]^{-1} \left\{ \begin{bmatrix} \bar{\theta}(\tilde{X}_t, t) \\ \tilde{h}(\tilde{X}_t, t) \end{bmatrix} - \partial_t \Lambda_t + \begin{bmatrix} \bar{\beta}(\tilde{X}_t, t) & \bar{\Lambda}_t \\ 0 & \end{bmatrix} \right\} \text{ on } O^x, \quad (3.5)$$

$$\phi^{x,s}(y, t) = \phi_{(\hat{\Lambda}, \bar{\kappa}, \bar{\beta}, \bar{\theta}, \tilde{h})}(y, t) = \Lambda_t^{-1} \left(\begin{bmatrix} \bar{c}_s(t) + G(t)(U_{s,t}^{-1} \tilde{X})y \\ \tilde{X}_t \end{bmatrix} \right) \quad (3.6)$$

on $\mathcal{N}^x = \left\{ (y, t) : \begin{bmatrix} \bar{c}_s(t) + G(t)U_{s,t}^{-1} \tilde{X}y \\ \tilde{X}_t \end{bmatrix} \in \Lambda_t(O^{x,s}) \right\}$. Finally, if $\check{\pi}, \check{\Lambda}$ and $\check{\kappa}$

also satisfies P0–P3, then there exist $\check{\beta}, \check{\theta}, \check{h}$ such that $(\check{\Lambda}, \check{\kappa}, \check{\beta}, \check{\theta}, \check{h}, \check{\pi}) \in \mathcal{P}$, $b_{(\check{\Lambda}, \check{\kappa}, \check{\beta}, \check{\theta}, \check{h}, \check{\pi})} = b_{(\hat{\Lambda}, \bar{\kappa}, \bar{\beta}, \bar{\theta}, \tilde{h})}$, and $\phi_{(\check{\Lambda}, \check{\kappa}, \check{\beta}, \check{\theta}, \check{h}, \check{\pi})} = \phi_{(\hat{\Lambda}, \bar{\kappa}, \bar{\beta}, \bar{\theta}, \tilde{h})}$.

Remark 6. For the sake of brevity in the examples below, we will just give local diffeomorphisms satisfying P3) above. However, as is shown in our companion paper Kouritzin (2016), it is often possible to solve for them using the technique used in the proof of Proposition 1 herein.

Remark 7. To illustrate the need of the final statement of Theorem 3, we take for example, $\sigma(x) = x \in \mathbb{R}^p$. Then, any $L \in C^1(\mathbb{R}^p)$ depending on $x_2/x_1, \dots, x_p/x_1$ satisfies $(\nabla L)\sigma = 0$. Therefore, $\hat{\Lambda}$ and hence the parameter set is not unique but we can create the same b, ϕ from any consistent $\bar{\kappa}, \hat{\Lambda}$.

3.1. One Dimensional Case. Suppose $d = p = r = 1$, $D \subset \mathbb{R}$ and $x \in D$. Then, $\bar{\kappa}, \tilde{h}$ do not exist and $\bar{\beta}, \bar{\theta}$ only depend on t . Moreover, $U_{s,t} = T_{s,t} = e^{-\int_s^t \bar{\beta}(u)du}$, $\bar{c}_s(t) = T_{s,t}^{-1}\bar{\Lambda}_s(x) + T_{s,t}^{-1} \int_s^t T_{s,u} \bar{\theta}(u)du$ and the diffeomorphism can be taken as $\Lambda_t(\varphi) = \int \frac{1}{\sigma(\varphi,t)} d\varphi$. One then finds by (1.4,3.3,3.5,3.6) that the corresponding diffusion drift b and explicit solutions are

$$b(\varphi, t) = \sigma(\varphi, t) \{ \bar{\theta}(t) + \bar{\beta}(t) \Lambda_t(\varphi) - \partial_t \Lambda_t \} + \frac{1}{2} \sigma(\varphi, t) \partial_\varphi \sigma(\varphi, t) \quad (3.7)$$

$$X_t = \Lambda_t^{-1} \left[\left\{ \Lambda_s(x) + \int_s^t T_{s,u} \bar{\theta}(u) du + \int_s^t T_{s,u} dW_u \right\} / T_{s,t} \right]. \quad (3.8)$$

Example 4 (Time-varying Cox-Ingersoll-Ross model). Suppose $\bar{\theta}, \bar{\beta}$ and continuously differentiable $s(t) > 0$ are chosen and $\sigma(\varphi, t) = s(t)\sqrt{\varphi}$. Then, $\Lambda_t(\varphi) = \frac{2\sqrt{\varphi}}{s(t)}$, $\Lambda_t^{-1}(z) = \left(\frac{zs(t)}{2} \right)^2$ and the possible Itô drifts are

$$b(\varphi, t) = \bar{\theta}(t)s(t)\sqrt{\varphi} + 2 \left(\bar{\beta}(t) + \frac{\dot{s}(t)}{s(t)} \right) \varphi + \frac{s^2(t)}{4}.$$

The explicit solutions are then

$$\begin{aligned} X_t^{x,s} &= \left| \frac{s(t)}{s(s)} e^{\int_s^t \bar{\beta}(v)dv} \sqrt{x} \right. \\ &\quad \left. + \frac{s(t)}{2} \left\{ \int_s^t e^{\int_s^u \bar{\beta}(v)dv} \bar{\theta}(u) du + \int_s^t e^{\int_s^u \bar{\beta}(v)dv} dW_u \right\} \right|^2. \end{aligned} \quad (3.9)$$

In the case $s(t) = \sigma, \bar{\theta}$ and $\bar{\beta}$ are taken constant, we get

$$X_t^{x,s} = \frac{1}{4} \left\{ 2e^{\bar{\beta}(t-s)} \sqrt{x} + \frac{\bar{\theta}\sigma}{\bar{\beta}} (e^{\bar{\beta}(t-s)} - 1) + \sigma \int_s^t e^{\bar{\beta}(t-u)} dW_u \right\}^2$$

solves

$$dX_t^{x,s} = \left(\sigma^2/4 + 2\bar{\beta}X_t^{x,s} + \sigma\bar{\theta}\sqrt{X_t^{x,s}} \right) dt + \sigma\sqrt{X_t^{x,s}} dW_t, \quad X_s = x$$

as long as $X_t^{x,s} > 0$. This solves the usual CIR model

$$dX_t = \alpha(\beta - X_t) dt + \sigma\sqrt{X_t} dW_t. \quad (3.10)$$

when $\bar{\theta} = 0$, $\alpha = 2\bar{\beta}$, $\beta = \sigma^2/(8\bar{\beta})$. Now, set $Y_t = \sqrt{X_t}$, where X solves (3.10) with $\sigma^2 = 4\alpha\beta$, and $\tau = \inf\{t > 0; X_t = 0\}$. It is well known that

$P(\tau < \infty) = 1$. Then,

$$\begin{aligned} dY_t &= \frac{1}{8Y_t} (4\alpha\beta - \sigma^2) dt - \frac{\alpha}{2} Y_t dt + \frac{\sigma}{2} dW_t \\ &= -\frac{\alpha}{2} Y_t dt + \frac{\sigma}{2} dW_t, \end{aligned} \quad (3.11)$$

by Itô's formula. However, since (3.11) defines a Gaussian process and Y must be non-negative, one cannot have Y_t defined by (3.11) unless $t < \tau$. This explains why we first look for explicit local solutions.

3.2. Square Non-Singular Case. Suppose that $d = p = r$, $\sigma = \sigma(\varphi, t)$ is a $d \times d$ non-singular continuously-differentiable matrix satisfying (2.12), $D \subset \mathbb{R}^p$ and $x \in D$. Again, we apply Theorem 3 and find $\bar{\kappa}, \tilde{h}$ do not exist while $\bar{\beta}, \bar{\theta}$ only depend on t . Also, there is a local diffeomorphism $\hat{\Lambda} = \begin{pmatrix} \Lambda_t \\ t \end{pmatrix}$ such that $\nabla_\varphi \Lambda_t(\varphi) = [\sigma(\varphi, t)]^{-1}$, and all explicit solutions are of the form $\phi^{x,s}(t, y) = \Lambda_t^{-1}(\bar{c}_s(t) + U_{s,t}^{-1}y)$, where

$$U_{s,t} = - \int_s^t U_{s,u} \bar{\beta}(u) du + I \text{ and } \bar{c}_s(t) = U_{s,t}^{-1} \left\{ \Lambda_s(x) + \int_s^t U_{s,u} \bar{\theta}(u) du \right\}$$

for some $\bar{\theta} \in C([0, T]; \mathbb{R}^d)$ and $\bar{\beta} \in C^1([0, T], \mathbb{R}^{d \times d})$. The resulting drift is

$$b(\varphi, t) = \sigma(\varphi, t) \left\{ \bar{\theta}(t) + \bar{\beta}(t) \Lambda_t(\varphi) - \partial_t \Lambda_t(\varphi) \right\} + \frac{1}{2} \sum_{j=1}^d (\nabla_\varphi \sigma_j(\varphi, t)) \sigma_j(\varphi, t).$$

Example 5. Geometric Brownian motions: Take $\sigma_{ij}(\varphi) = \varphi_i \gamma_{ij}$ with γ non-singular and $D = (0, \infty)^d$. Then, σ satisfies the commutation condition (2.12) since $[(\nabla_\varphi \sigma_j) \sigma_k]_i = \varphi_i \gamma_{ij} \gamma_{ik}$, and the diffeomorphism can be chosen as $\Lambda(\varphi) =$

$$\Lambda_t(\varphi) = \gamma^{-1} \begin{bmatrix} \log \varphi_1 \\ \vdots \\ \log \varphi_d \end{bmatrix}. \quad \Lambda \text{'s image is } \mathbb{R}^d, \text{ so } \Lambda^{-1}(z) = \begin{bmatrix} e^{(\gamma z)_1} \\ \vdots \\ e^{(\gamma z)_d} \end{bmatrix} \text{ is defined everywhere and } \phi_i^{x,s}(y, t) = \exp [\gamma \{ \bar{c}_s(t) + U_{s,t}^{-1} y \}]_i. \text{ The possible drifts satisfy}$$

$$b_i(\varphi, t) = \varphi_i \left\{ \alpha_i(t) - \sum_{j=1}^d B_{ij}(t) \log \varphi_j \right\},$$

for $1 \leq i \leq d$, where $B(t) = \gamma \bar{\beta}(t) \gamma^{-1}$, and $\alpha_i(t) = \frac{1}{2} [\gamma \gamma^\top]_{ii} + [\gamma \bar{\theta}(t)]_i$.

Example 6. Diffeomorphism example: In the previous examples, we started with σ . Suppose instead we had a diffeomorphism

$$\Lambda(\varphi_1, \varphi_2) = \Lambda_t(\varphi_1, \varphi_2) = \begin{bmatrix} \frac{\pi}{2} + \arcsin(\log \varphi_1 \varphi_2 - 1) \\ \frac{\pi}{2} + \arcsin(\frac{2\varphi_2}{\varphi_1} - 1) \end{bmatrix}$$

on $1 < \varphi_1\varphi_2 < e$, $\varphi_2 \leq \varphi_1$. Then, the possible full rank σ 's satisfy $\sigma = (\nabla_\varphi \Lambda)^{-1}$ i.e.

$$\sigma(\varphi_1, \varphi_2) = \begin{pmatrix} \frac{\varphi_1}{2} \sqrt{2 \log \varphi_1 \varphi_2 - (\log \varphi_1 \varphi_2)^2} & -\frac{\varphi_1}{2\varphi_2} \sqrt{\varphi_2(\varphi_1 - \varphi_2)} \\ \frac{\varphi_2}{2} \sqrt{2 \log \varphi_1 \varphi_2 - (\log \varphi_1 \varphi_2)^2} & -\frac{1}{2} \sqrt{\varphi_2(\varphi_1 - \varphi_2)} \end{pmatrix} \quad (3.12)$$

so $(\nabla \Lambda)\sigma = I_2$ and σ satisfies (2.12) by Lemma 1 ii). The possible Stratonovich (time-dependent) drifts $h(\varphi_1, \varphi_2, t)$ are

$$\sigma(\varphi_1, \varphi_2) \begin{pmatrix} \bar{\theta}_1(t) + \bar{\beta}_{11}(t)(\frac{\pi}{2} + \arcsin(\log \varphi_1 \varphi_2 - 1)) - \bar{\beta}_{12}(t)(\frac{\pi}{2} + \arcsin(\frac{2\varphi_2}{\varphi_1} - 1)) \\ \bar{\theta}_2(t) + \bar{\beta}_{21}(t)(\frac{\pi}{2} + \arcsin(\log \varphi_1 \varphi_2 - 1)) - \bar{\beta}_{22}(t)(\frac{\pi}{2} + \arcsin(\frac{2\varphi_2}{\varphi_1} - 1)) \end{pmatrix} \quad (3.13)$$

while $U_{s,t}, \bar{c}_s$ satisfy the equations at the start of Subsection 3.2.

3.3. Non-Square Case. Our most important example is probably the *Extended Heston* model of our companion paper Kouritzin (2016). It is non-square. However, we provide a second interesting non-square example herein.

Example 7 (Heisenberg group). Let $\bar{x} \in \mathbb{R}^d$ and $\tilde{x} \in \mathbb{R}$ be the components of the starting point, $A = A(t)$ be a $\mathbb{R}^{d \times d}$ continuously differentiable matrix function and $\sigma(\varphi, t) = \sigma(\xi, z, t) = \begin{bmatrix} I_d \\ (A(t)\xi)^\top \end{bmatrix}$, where $\xi \in \mathbb{R}^d$, $z \in \mathbb{R}$. Then, σ has rank $r = d$. The solution to $dX_t = \sigma(X_t, t)dW_t$ is known as the Brownian motion on the Heisenberg group. Moreover,

$$(\nabla_\varphi \sigma_j) \sigma_k - (\nabla_\varphi \sigma_k) \sigma_j = \begin{bmatrix} 0 \\ A_{jk} - A_{kj} \end{bmatrix}.$$

Therefore, (2.12) holds true if and only if A is symmetric. In this case, one can solve for an explicit solution for an arbitrary starting point (\bar{x}, \tilde{x}, s) .

The diffeomorphism $\hat{\Lambda}(\xi, z, t) = \begin{bmatrix} \Lambda_t(\xi, z) \\ t \end{bmatrix}$ is solved $\Lambda_t(\xi, z) = \begin{bmatrix} \xi \\ g \end{bmatrix}$ with $g(\xi, z, t) = z - \frac{1}{2}\xi^\top A(t)\xi$ following the proof of Proposition 1 in the Appendix (see Kouritzin (2016) for details on a more involved example). Hence, $\pi = I_d$,

$\hat{\sigma} = \begin{bmatrix} I_d \\ 0 \end{bmatrix}$, $\bar{\kappa}$ does not exist so $G(t) = I_d$ and $[\nabla \Lambda_t]^{-1} = \begin{bmatrix} I_d & 0 \\ \xi^\top A(t) & 1 \end{bmatrix}$. Now,

we can take any functions $\bar{\theta} \in \mathbb{R}^d$, $\bar{\beta} \in \mathbb{R}^{d \times d}$, $\tilde{h} \in \mathbb{R}$ satisfying the differentiability conditions in Definition 3 and let $\tilde{X}_t, U_{s,t}\tilde{X}, \bar{c}_s(t)$ satisfy:

$$\begin{aligned} \frac{d}{dt} \tilde{X}_t &= \tilde{h}(\tilde{X}_t, t) \text{ s.t. } \tilde{X}_s = \tilde{x} - \frac{1}{2} \bar{x}^\top A(s) \bar{x} \\ \frac{d}{du} U_{s,u} \tilde{X} &= -(U_{s,u} \tilde{X}) \bar{\beta}(\tilde{X}_u, u) \text{ s.t. } U_{s,s} \tilde{X} = I_d \\ \bar{c}_s(t) &= U_{s,t}^{-1} \left\{ \bar{x} + \int_0^t U_{s,u} \bar{\theta}(\tilde{X}_u, u) du \right\}. \end{aligned}$$

From Theorem 3 and (1.4), drift b must be of the (quadratic) form

$$b(\xi, z, t) = \left[\tilde{h}(\tilde{X}_t, t) + \xi^\top A(t) \bar{\theta}(\tilde{X}_t, t) - \xi^\top A(t) \bar{\beta}(\tilde{X}_t, t) \xi + \frac{1}{2} \xi^\top \frac{d}{dt} A(t) \xi + \frac{1}{2} \text{Tr}\{A(t)\} \right]$$

for some $\bar{\theta}$, $\bar{\beta}$, \tilde{h} . Finally, the corresponding ϕ is given by

$$\phi(y, t) = \left[\tilde{X}_t + \frac{1}{2} (\bar{c}_s(t) + (U_{s,t}^{-1} \tilde{X})y)^\top A(t) (\bar{c}_s(t) + (U_{s,t}^{-1} \tilde{X})y) \right].$$

4. COMPARISON WITH THE WORKS OF YAMATO AND KUNITA

Now, we compare our existence results to those appearing in Yamato (1979) and Kunita (1984). In Section III.3 of Kunita's treatise, he considers representations of time-homogeneous Fisk-Stratonovich equations

$$dX_t^x = h(X_t^x)dt + \sigma(X_t^x) \bullet dW_t \quad (4.1)$$

in terms of the flows generated by the vector fields

$$\mathfrak{X}_0(y) = \sum_{i=1}^p h_i(y) \frac{\partial}{\partial y_i} \text{ and } \mathfrak{X}_k(y) = \sum_{i=1}^p \sigma_{ik}(y) \frac{\partial}{\partial y_i}, k = 1, \dots, d, \quad (4.2)$$

under conditions imposed on the Lie algebra $L_0(\mathfrak{X}_0, \mathfrak{X}_1, \dots, \mathfrak{X}_d)$ generated by \mathfrak{X}_k , $0 \leq k \leq d$. In the special case where these vector fields commute, i.e. the Lie bracket $[\mathfrak{X}_k, \mathfrak{X}_j] = 0$ for each $j, k = 0, \dots, d$, and the coefficients h_i , σ_{ik} are respectively in C_α^3 , C_α^4 (the locally four times continuously differentiable functions whose fourth derivative is α -Hölder continuous), his work gives rise to the composition formula

$$\begin{aligned} (X_t^x)_i &= \text{Exp}(t\mathfrak{X}_0) \circ \text{Exp}(W_t^1 \mathfrak{X}_1) \circ \dots \circ \text{Exp}(W_t^d \mathfrak{X}_d) \circ \chi_i(x), \\ &= \phi_i(W_t, t) \end{aligned} \quad (4.3)$$

locally. Here, χ_i is the function taking x to its i^{th} component and $\text{Exp}(u\mathfrak{X}_k)$ is the one parameter group of transformations generated by vector field \mathfrak{X}_k , i.e. the unique solution to

$$\frac{d}{du}(f \circ \varphi_u) = \mathfrak{X}_k f(\varphi_u), \quad \varphi_0 = x \quad \forall f \in C^\infty. \quad (4.4)$$

In fact, to use (4.3), one must solve (4.4) for $k = 0, \dots, d$ and $f = \chi_i$, $i = 1, \dots, d$. Kunita also goes beyond commutability, even surpassing Yamato (1979) in generality by considering the situation where $L_0(\mathfrak{X}_0, \dots, \mathfrak{X}_d)$ is only solvable, but the expression replacing (4.3) necessarily becomes more unwieldy.

Our characterization of ϕ provided by Theorem 3 provides an alternative to (4.3) that is more amenable to direct calculation. Corollary 1 (to follow) supplies a converse to (4.3) in the sense that if $X_t^{x,s}$ were to have such a functional representation $\phi^{x,s}(W_t, t)$ in terms of Brownian motions only, then the vector fields must commute. This was previously established in Theorem 4.1 of Yamato (1979) under C^∞ conditions on both ϕ and the coefficients.

The other advantages of our representations over Kunita's results are:

- We allow time dependent vector fields.
- We decrease the regularity assumptions by imposing weaker differentiability on h and on σ when r is small. The looser regularity on the coefficients requires eschewing Fisk-Stratonovich equations in favour of Itô processes.
- We remove the nilpotency assumptions (for our representations).

To validate the final claim, we take $p = 2$, $d = 1$,

$$\mathfrak{X}_0 = \{\bar{\theta}(x_2) - B(x_2)x_1\}\partial_{x_1} + \tilde{\theta}(x_2)\partial_{x_2},$$

and $\mathfrak{X}_1 = \partial_{x_1}$. Then $[\mathfrak{X}_0, \mathfrak{X}_1] = B\partial_{x_1}$. Moreover, if $\mathfrak{X}_k = [\mathfrak{X}_0, \mathfrak{X}_{k-1}]$, $k \geq 2$, then $\mathfrak{X}_k = a_k(x_2)\partial_{x_1}$, where $a_{k+1} = \tilde{\theta}(\partial_{x_2}a_k) + Ba_k$, $k \geq 1$ and $a_1 = 1$. In general, the a_k 's will not vanish and thereby the Lie algebra contains an infinite number of linearly independent vector fields. This algebra is solvable but is not nilpotent.

Using Theorem 1, we can also give the converse to Kunita's result, Example III.3.5 in Kunita (1984), that is valid under the mild regularity on b, σ, h given at the beginning of the section.

Corollary 1. *Suppose that there exists a domain \tilde{D} such that the coefficients σ and h are time-homogeneous and Fisk-Stratonovich acceptable on $\tilde{D}_T = \tilde{D} \times (0, T)$. Further, assume that the solution to the Fisk-Stratonovich equation (4.1) has a unique local solution*

$$(X_t^x)_i = \text{Exp}(t\mathfrak{X}_0) \circ \text{Exp}(W_t^1\mathfrak{X}_1) \circ \cdots \circ \text{Exp}(W_t^d\mathfrak{X}_d) \circ \chi_i(x)$$

on $0 \leq t < \tau_x$ for some positive stopping time τ_x and each $x \in \tilde{D}$, where \mathfrak{X}_k , $k = 0, 1, \dots, d$ are the vector fields defined in (4.2). Then,

$$[\mathfrak{X}_k, \mathfrak{X}_j] = 0 \text{ on } \tilde{D} \text{ for each } j, k = 0, \dots, d.$$

Proof. We find that $X_t^x = \phi(Y_t, t)$ with $U_{s,t} = I$ so it follows from Theorem 1 that $\sigma A = 0$. The condition $[\mathcal{X}_k, \mathcal{X}_j] = 0$ then follows from (2.12, 2.13). \square

5. PROOFS OF THE MAIN RESULTS

We note that b, σ are Lipschitz on any compact, convex subset of D_T by our C^1 -conditions and use the proof of Kunita (1984)[Theorem II.5.2] for uniqueness of (strong) local solutions to the SDE until they leave such a compact subset.

5.1. Proof of Theorem 1 a) is equivalent to b).

Proof. Using (2.1) and Itô's formula for $X_t = \phi(Y_t, t)$, one finds that for any $1 \leq i \leq p$,

$$\begin{aligned} d(X_t)_i &= \sum_{m=1}^d \sum_{j=1}^d \partial_{y_m} \phi_i(Y_t, t) (U_{s,t} \phi)_{mj} dW_t^j \\ &+ \left[\partial_t \phi_i(Y_t, t) + \frac{1}{2} \sum_{j=1}^d \sum_{k=1}^d \partial_{y_j} \partial_{y_k} \phi_i(Y_t, t) (U_{s,t} \phi (U_{s,t} \phi)^\top)_{jk} \right] dt. \end{aligned} \quad (5.1)$$

Now, starting with b) implies a) and using (2.7,2.8) on (5.1), we find

$$\begin{aligned} d(X_t)_i &= \sigma_i(\phi(Y_t, t), t) dW_t + h_i(\phi(Y_t, t), t) dt \\ &+ \frac{1}{2} \sum_{j=1}^d \sum_{k=1}^d \partial_{y_j} \partial_{y_k} \phi_i(Y_t, t) (U_{s,t} \phi (U_{s,t} \phi)^\top)_{jk} dt. \end{aligned} \quad (5.2)$$

Moreover,

$$\partial_{y_m} \{\sigma_{ij}(\phi, t)\} = \sum_{n=1}^p \{\partial_{\varphi_n} \sigma_{ij}\}(\phi, t) \partial_{y_m} \phi_n$$

and if (2.7) is true, one obtains

$$\partial_{y_m} \{\sigma_{ij}(\phi, t)\} = \sum_{l=1}^d \partial_{y_m} \partial_{y_l} \phi_i (U_{s,t} \phi)_{lj}.$$

Abbreviating notation $U_{mk}(\phi, t) = (U_{s,t} \phi)_{mk}$, multiplying the last two equalities by U_{mk} , summing over m and using (2.7) again, one finds that

$$\sum_{n=1}^p \{\partial_{\varphi_n} \sigma_{ij}\}(\phi, t) \sigma_{nk}(\phi, t) = \sum_{m=1}^d \sum_{l=1}^d \partial_{y_m} \partial_{y_l} \phi_i U_{lj}(\phi, t) U_{mk}(\phi, t), \quad (5.3)$$

and, taking $k = j$ and summing over j , one has that

$$\sum_{j=1}^d \{\nabla_{\varphi} \sigma_j\}(\phi, t) \sigma_j(\phi, t) = \sum_{l=1}^d \sum_{m=1}^d (U(\phi, t) U^\top(\phi, t))_{lm} \partial_{y_m} \partial_{y_l} \phi. \quad (5.4)$$

Therefore, if (2.7,2.8,2.2) are satisfied, then clearly X_t is a local strong solution to (2.10) by (1.4). Moreover, letting $t \searrow s$, we find by (2.7,2.8,2.2) that

$$\sigma(x, s) = \nabla_y \phi^{x,s}(0, s) \quad \text{and} \quad h(x, s) = \partial_t \phi^{x,s}(0, s)$$

so $\sigma, h \in C^1$ by the last part of Definition 1.

To show a) implies b), we suppose X_t is a strong solution to (2.10) on $(s, \tau^{x,s})$. Then, since continuous finite-variation martingales are constant, the (continuous) Itô process $\phi(Y_t, t)$ from (5.1) matches (2.10) if and only if

$$\sigma_{ij}(\phi, t) = \sum_{m=1}^d \partial_{y_m} \phi_i (U_{s,t} \phi)_{mj} \quad \forall 1 \leq i \leq p, \quad 1 \leq j \leq d, \quad (5.5)$$

and

$$b_i(\phi, t) = \partial_t \phi_i + \frac{1}{2} \sum_{j=1}^d \sum_{k=1}^d \partial_{y_j} \partial_{y_k} \phi_i (U_{s,t} \phi (U_{s,t} \phi)^\top)_{jk} \quad \forall 1 \leq i \leq p \quad (5.6)$$

for all $t \in (s, \tau^{x,s})$. Rewriting (5.5) in matrix form, one finds

$$\sigma(\phi(Y_t, t), t) = \{\nabla_y \phi(Y_t, t)\} U_{s,t} \phi, \quad (5.7)$$

and (2.7) is true. Now, we can use (5.4) (which was just shown to be a consequence of (2.7)) to find (5.6) is equivalent to

$$\partial_t \phi = b(\phi, t) - \frac{1}{2} \sum_{k=1}^d \{\nabla_x \sigma_k\}(\phi, t) \sigma_k(\phi, t) = h(\phi, t), \quad (5.8)$$

using (1.4). Now, (2.8) follows by continuity and (2.6). Letting $t \searrow s$ in (5.7) and (5.8), one finds

$$\sigma(x, s) = \nabla_y \phi^{x,s}(0, s) \quad \text{and} \quad h(x, s) = \partial_t \phi^{x,s}(0, s)$$

so the last part of Definition 1 follows from the C^1 property of h, σ . \square

5.2. Proof of Theorem 1 b) is equivalent to c). Idea: Below we show that the existence of a representation without the commutator conditions leads to a contradiction and the commutator conditions yield a representation.

Proof. By exactness of differential 1-forms, the existence of our function $\phi^{x,s}$ satisfying ((2.7), (2.8) and (2.2)) is equivalent to the following two conditions:

$$\partial_{y_j} \{\sigma(\phi, t)(U_{s,t}^{-1} \phi)_k\} = \partial_{y_k} \{\sigma(\phi, t)(U_{s,t}^{-1} \phi)_j\} \quad (5.9)$$

and

$$\frac{d}{dt} \{\sigma(\phi, t)(U_{s,t}^{-1} \phi)_k\} = \partial_{y_k} h(\phi, t). \quad (5.10)$$

We show (5.9) and (5.10) for all starting points (x, s) are equivalent to (2.12) and (2.13) respectively.

Step 1: Show that (2.12) implies (5.9) (under (2.7)).

It follows by (2.7) and $\mathcal{C}_2, \mathcal{C}_3$ that

$$\begin{aligned} & \partial_{y_j} \{\sigma(\phi, t)(U_{s,t}^{-1} \phi)_k\} \\ &= \sum_m \{\partial_{y_j} \sigma_m(\phi, t)\} (U_{s,t}^{-1} \phi)_{mk} \\ &= \sum_m \nabla_\phi \sigma_m(\phi, t) \sigma(\phi, t)(U_{s,t}^{-1} \phi)_j (U_{s,t}^{-1} \phi)_{mk} \\ &= \sum_m \sum_n \nabla_\phi \sigma_m(\phi, t) \sigma_n(\phi, t)(U_{s,t}^{-1} \phi)_{nj} (U_{s,t}^{-1} \phi)_{mk} \end{aligned} \quad (5.11)$$

and similarly

$$\begin{aligned} & \partial_{y_k} \{ \sigma(\phi, t) (U_{s,t}^{-1} \phi)_j \} \\ &= \sum_n \sum_m \nabla_\phi \sigma_n(\phi, t) \sigma_m(\phi, t) (U_{s,t}^{-1} \phi)_{mk} (U_{s,t}^{-1} \phi)_{nj} \end{aligned} \quad (5.12)$$

Hence, (5.9) holds when (2.12) holds.

Step 2: Show that (5.9) implies (2.12) (under (2.7)).

Letting $t \searrow s$ in (5.11) and (5.12), one finds by (5.9) that for all $1 \leq j, k \leq d$,

$$\begin{aligned} & \sum_m \sum_n \nabla_x \sigma_m(x, s) \sigma_n(x, s) (U_{s,s}^{-1} \phi)_{nj} (U_{s,s}^{-1} \phi)_{mk} \\ &= \lim_{t \searrow s} \partial_{y_j} \{ \sigma(\phi, t) (U_{s,t}^{-1} \phi)_k \} \\ &= \lim_{t \searrow s} \partial_{y_k} \{ \sigma(\phi, t) (U_{s,t}^{-1} \phi)_j \} \\ &= \sum_m \sum_n \nabla_x \sigma_n(x, s) \sigma_m(x, s) (U_{s,s}^{-1} \phi)_{nj} (U_{s,s}^{-1} \phi)_{mk}. \end{aligned} \quad (5.13)$$

However, $U_{s,s}^{-1} \phi = I$ so we have that

$$(\nabla_x \sigma_q)(x, s) \sigma_p(x, s) = (\nabla_x \sigma_p)(x, s) \sigma_q(x, s).$$

Hence, (2.12) holds when (5.9) does.

Step 3: Show that (5.10) implies (2.13) (under (2.7, 2.8)).

One gets by (2.7) that

$$\partial_{y_k} h(\phi, t) = \nabla_\phi h(\phi, t) \partial_{y_k} \phi(y, t) = \nabla_\phi h(\phi, t) \sigma(\phi, t) (U_{s,t}^{-1} \phi)_k \quad (5.14)$$

and by (5.10), (2.8) that

$$\begin{aligned} \partial_{y_k} h(\phi, t) &= \frac{d}{dt} \{ \sigma(\phi, t) (U_{s,t}^{-1} \phi)_k \} \\ &= \sum_m \nabla_\phi \sigma_m(\phi, t) h(\phi, t) (U_{s,t}^{-1} \phi)_{mk} + \partial_t \sigma(\phi, t) (U_{s,t}^{-1} \phi)_k \\ &\quad - \sigma(\phi, t) U_{s,t}^{-1} \phi \sum_m \frac{d}{dt} (U_{s,t} \phi)_m (U_{s,t}^{-1} \phi)_{mk}. \end{aligned} \quad (5.15)$$

Combining these equations, multiplying by $(U_{s,t} \phi)_{kn}$ and summing, we get

$$\begin{aligned} \nabla_\phi h(\phi, t) \sigma_n(\phi, t) &= \nabla_\phi \sigma_n(\phi, t) h(\phi, t) + \partial_t \sigma_n(\phi, t) \\ &\quad - \sigma(\phi, t) U_{s,t}^{-1} \phi^{x,s} \frac{d}{dt} (U_{s,t} \phi^{x,s})_n \end{aligned} \quad (5.16)$$

so, letting $t \searrow s$ and using (2.11, 2.2), one arrives at (2.13).

Step 4: Show that (2.13) implies (5.10) (under (2.7, 2.8)).

Using (2.8) and (2.4), we get that

$$\begin{aligned}
& \frac{d}{dt} \{ \sigma(\phi, t) (U_{s,t}^{-1} \phi)_k \} \\
&= \sum_n \left[\{ \nabla_\phi \sigma_n(\phi, t) \} h(\phi, t) + \{ \partial_t \sigma_n(\phi, t) \} \right] (U_{s,t}^{-1} \phi)_{nk} \\
&\quad - \sum_n \sigma(\phi, t) U_{s,t}^{-1} \phi \left(\frac{d}{dt} (U_{s,t} \phi)_n \right) (U_{s,t}^{-1} \phi)_{nk} \\
&= \sum_n \left[\{ \nabla_\phi \sigma_n(\phi, t) \} h(\phi, t) + \partial_t \sigma_n(\phi, t) \right. \\
&\quad \left. - \sigma(\phi, t) \left(\frac{d}{dt} U_{u,t} \phi^{\phi_u, u} \right)_n \Big|_{u=t} \right] (U_{s,t}^{-1} \phi)_{nk},
\end{aligned}$$

where ϕ_u is short for $\phi^{x,s}(y_u, u)$. Hence, by (2.11), (2.13) applied at $\varphi = \phi$ and (2.7)

$$\begin{aligned}
\frac{d}{dt} \{ \sigma(\phi, t) (U_{s,t}^{-1} \phi)_k \} &= \{ \nabla_\phi h(\phi, t) \} \sigma(\phi, t) (U_{s,t}^{-1} \phi)_k \\
&= \partial_{y_k} h(\phi, t)
\end{aligned}$$

and we have (5.10). \square

5.3. Proof of Proposition 1. Our methods are motivated in part by Brickell and Clark (1970)[Propositions 8.3.2 and 11.5.2].

We let

$$(q, D_T^2) = \begin{cases} (p+1, D \times (-T, T)) & \text{if } \sigma \text{ or } h \text{ depend on } t \\ (p, D) & \text{otherwise} \end{cases},$$

take $\sigma_{p+1} = 0$ if $q > p$, set $\partial_t \sigma(x, t) = \partial_t \sigma(x, 0)$, $\partial_{x_i} \sigma(x, t) = \partial_{x_i} \sigma(x, 0)$ for $t < 0$, $i = 1, 2, \dots, q$ and use exactness of the corresponding 1-form to extend σ uniquely to D_T^2 such that $\sigma \in C^1(D_T^2; \mathbb{R}^{q \times d})$. By reducing $T > 0$ if necessary, we can find a permutation π such that the first r columns of $\sigma^\pi = \sigma \pi$ are linearly independent on D_T^2 .

Proof. Fix $\hat{x} = (\hat{x}_1, \dots, \hat{x}_q) \in D_T$. The C^1 -diffeomorphism Λ will have form:

$$\Lambda = \Lambda^{r,1}, \text{ where } \Lambda^{i,1} = \Lambda^i \circ \Lambda^{i-1} \circ \dots \circ \Lambda^2 \circ \Lambda^1, \quad (5.17)$$

$$\Lambda^i = \sum_{j=1}^{i-1} x_j e_j + \begin{bmatrix} H^i(x_i, \dots, x_q) \\ L^i(x_i, \dots, x_q) \end{bmatrix} \text{ and } H^i(x_i, \dots, x_q) \in \mathbb{R}^{i-1}, \quad (5.18)$$

and $\hat{\sigma}_i$ will be defined as $\hat{\sigma}_i \doteq \{ (\nabla_\varphi \Lambda) \sigma_i^\pi \} \circ \Lambda^{-1}$. Here, Λ^i is a C^1 -diffeomorphism on a neighborhood $O^{\hat{x}^{i-1}}$ of $\hat{x}^{i-1} = \Lambda^{i-1,1}(\hat{x})$ so $\Lambda : O^{\hat{x}} \rightarrow \mathbb{R}^q$.

To construct Λ^i recursively starting with Λ^1 , we suppose $\hat{\sigma}_j = e_j$ for $j < i$ and

$$\alpha_i \doteq \{ \nabla \Lambda^{i-1,1} \sigma_i^\pi \} \circ (\Lambda^{i-1,1})^{-1} \quad (5.19)$$

does not depend upon x_1, \dots, x_{i-1} , which are certainly true when $i = 1$. Moreover, without loss of generality, we assume the i^{th} component of α_i satisfies $\alpha_{i,i} \neq 0$ (or else we change π by permuting columns i, \dots, d of σ^π). Set $\psi^i(x) = \theta(x_i - \hat{x}_i^{i-1}; x_1, \dots, x_{i-1}, \hat{x}_i^{i-1}, x_{i+1}, \dots, x_q)$, where θ satisfies $\frac{d}{dt}\theta(t; x) = \alpha_i(\theta(t; x))$, $\theta(0; x) = x$ for $t \in I^x$, an open interval containing 0, and x in a neighborhood containing \hat{x}^{i-1} . Then, $\partial_{x_i}\psi^i = \alpha_i(\psi^i)$. For $j \neq i$, we have $\partial_{x_j}\psi^i(x) = \partial_{x_j}\theta(x_i - \hat{x}_i^{i-1}; x_1, \dots, x_{i-1}, \hat{x}_i^{i-1}, x_{i+1}, \dots, x_q)$ and

$$\partial_t \partial_{x_j} \theta(t; x) = \partial_{x_j} \alpha_i(\theta(t; x)) \quad \text{s.t.} \quad \partial_{x_j} \theta(0, x) = e_j$$

so $\nabla \psi^i(\hat{x}^{i-1})$ has determinant $\alpha_{i,i}(\hat{x}^{i-1}) \neq 0$. Thus, ψ^i has inverse $\Lambda^i \in C^2(O^{\hat{x}^{i-1}}, \mathbb{R}^q)$ and $\nabla \Lambda^i = [\nabla \psi^i]^{-1}(\Lambda^i)$ on neighborhood $O^{\hat{x}^{i-1}}$ of \hat{x}^{i-1} by the Inverse Function Theorem. Hence, $\nabla \Lambda^i ((\Lambda^i)^{-1}) \nabla \psi^i = I$ and

$$\hat{\sigma}_i = \{\nabla \Lambda^i \alpha_i\} (\Lambda^i)^{-1} = e_i \in \mathbb{R}^q. \quad (5.20)$$

Moreover, Λ^i has the form (5.18) if ψ^i has similar form. ψ^i has this form by its definition as well as the facts α_i is locally Lipschitz and does not depend upon x_1, \dots, x_{i-1} . Next,

$$(\nabla \hat{\sigma}_j) \hat{\sigma}_k - (\nabla \hat{\sigma}_k) \hat{\sigma}_j = (\nabla \sigma_j^\pi) \sigma_k^\pi - (\nabla \sigma_k^\pi) \sigma_j^\pi = 0 \quad \forall \quad 1 \leq k, j \leq d \quad (5.21)$$

by Lemma 1. Now, since $\hat{\sigma}_k = e_k \in \mathbb{R}^q$ for $1 \leq k \leq i$, (5.21) implies

$$(\nabla \hat{\sigma}_j) e_k = (\nabla \hat{\sigma}_j) e_k - (\nabla e_k) \hat{\sigma}_j = 0 \quad \forall \quad 1 \leq k \leq i < j$$

on a neighborhood O of \hat{x} . Therefore, $\hat{\sigma}_j$ and (by a similar argument) α_{i+1} can not depend upon x_1, \dots, x_i so we can take $i = r$ by induction and

$$\hat{\sigma} = \{(\nabla \Lambda) \sigma^\pi\} \circ \Lambda^{-1} = \begin{pmatrix} I_r & \bar{\kappa} \\ 0 & \tilde{\kappa} \end{pmatrix} \in \mathbb{R}^{q \times d} \text{ on } \Lambda(O \cap D_T),$$

where $\bar{\kappa} \in \mathbb{R}^{r \times (d-r)}$ and $\tilde{\kappa} \in \mathbb{R}^{(q-r) \times (d-r)}$ do not depend on the variables x_1, \dots, x_r . Since $\hat{\sigma}$ has also rank r , it follows that $\tilde{\kappa} = 0$. \square

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